

ON THE GRADED QUOTIENTS OF THE SL(m, \mathbf{C})-REPRESENTATION ALGEBRAS OF GROUPS

TAKAO SATOH¹

ABSTRACT. In this paper, we consider certain descending filtrations of the SL(m, \mathbf{C})-representation algebras of free groups and free abelian groups. By using it, we introduce analogs of the Johnson homomorphisms of the automorphism groups of free groups. We show that the first homomorphisms are extended to the automorphism groups of free groups as crossed homomorphisms. Furthermore we show that the extended crossed homomorphisms induce Kawazumi's cocycles and Morita's cocycles. This works are generalization of our previous results [31] and [32] for the SL(2, \mathbf{C})-representation algebras.

1. INTRODUCTION

For any $m \geq 2$ and any group G , let $R^m(G)$ be the set $\text{Hom}(G, \text{SL}(m, \mathbf{C}))$ of all SL(m, \mathbf{C})-representations of G . Let $\mathcal{F}(R^m(G), \mathbf{C})$ be the set $\{\chi : R^m(G) \rightarrow \mathbf{C}\}$ of all complex-valued functions on $R^m(G)$. Then $\mathcal{F}(R^m(G), \mathbf{C})$ naturally has the \mathbf{C} -algebra structure coming from the pointwise sum and product. (See Subsection 3 for details.) For any $x \in G$ and any $1 \leq i, j \leq m$, we define an element $a_{ij}(x)$ of $\mathcal{F}(R^m(G), \mathbf{C})$ to be

$$(a_{ij}(x))(\rho) := (i, j)\text{-component of } \rho(x)$$

for any $\rho \in R^m(G)$. We call the map $a_{ij}(x)$ the (i, j) -component function of x , or simply a component function of x . Let $\mathfrak{R}_{\mathbf{Q}}^m(G)$ be the \mathbf{Q} -subalgebra of $\mathcal{F}(R^m(G), \mathbf{C})$ generated by all $a_{ij}(x)$ for $x \in G$ and $1 \leq i, j \leq m$. We call $\mathfrak{R}_{\mathbf{Q}}^m(G)$ the SL(m, \mathbf{C})-representation algebras of G over \mathbf{Q} . In this paper, we introduce a descending filtration of $\mathfrak{R}_{\mathbf{Q}}^m(G)$ consisting of Aut G -invariant ideals, and study the graded quotients of it.

To the best of our knowledge, the study of the algebra $\mathfrak{R}_{\mathbf{Q}}^m(G)$ has a not so long history. Classically, the \mathbf{Q} -subalgebra of $\mathfrak{R}_{\mathbf{Q}}^2(F_n)$ generated by characters of F_n was actively studied. For any $x \in F_n$, the map $\text{tr } x := a_{11}(x) + a_{22}(x)$ is called the Fricke character of x . Fricke and Klein [6] used the Fricke characters for the study of the classification of Riemann surfaces. In the 1970s, Horowitz [11] and [12] investigated several algebraic properties of the algebra of Fricke characters by using the combinatorial group theory. In 1980, Magnus [19] studied some relations among Fricke characters of free groups systematically. As is found in Acuna-Maria-Montesinos's paper [1], today Magnus's research has been developed to the study of the SL(2, \mathbf{C})-character varieties of free groups by quite many authors. Let $\mathfrak{X}_{\mathbf{Q}}^2(F_n)$ be the \mathbf{Q} -subalgebra of $\mathfrak{R}_{\mathbf{Q}}^2(F_n)$ generated by all $\text{tr } x$ for $x \in F_n$. The algebra $\mathfrak{X}_{\mathbf{Q}}^2(F_n)$ is called the algebra of Fricke characters of F_n . Let \mathfrak{C} be the ideal of $\mathfrak{X}_{\mathbf{Q}}^2(F_n)$ generated by $\text{tr } x - 2$ for any $x \in F_n$. In our previous

2010 *Mathematics Subject Classification.* 20F28 (Primary), 20G05, 13A15 (Secondary).

Key words and phrases. SL(m, \mathbf{C})-representation algebras, Automorphism groups of free groups, Johnson homomorphisms.

¹e-address: takao@rs.tus.ac.jp

papers [9] and [30], we considered an application of the theory of the Johnson homomorphisms of $\text{Aut } F_n$ by using the Fricke characters. In particular, we determined the structure of the graded quotients $\text{gr}^k(\mathfrak{C}) := \mathfrak{C}^k / \mathfrak{C}^{k+1}$ for $1 \leq k \leq 2$, introduced analogs of the Johnson homomorphisms, and showed that the first homomorphism extends to $\text{Aut } F_n$ as a crossed homomorphism.

We briefly review the history of the Johnson homomorphisms. In 1965, Andreadakis [2] introduced a certain descending central filtration of $\text{Aut } F_n$ by using the natural action of $\text{Aut } F_n$ on the nilpotent quotients of F_n . We call this filtration the Andreadakis-Johnson filtration of $\text{Aut } F_n$. (For definitions, see Section 4.1.) In the 1980s, Johnson studied such filtration for mapping class groups of surfaces in order to investigate the group structure of the Torelli groups in a series of works [13], [14], [15] and [16]. In particular, he determined the abelianization of the Torelli group by introducing a certain homomorphism. Today, his homomorphism is called the first Johnson homomorphism, and it is generalized to higher degrees. Over the last two decades, the Johnson homomorphisms of the mapping class groups have been actively studied from various viewpoints by many authors including Morita [21], Hain [8] and others.

The Johnson homomorphisms are naturally defined for $\text{Aut } F_n$:

$$\tilde{\tau}_k : \mathcal{A}_{F_n}(k) \hookrightarrow \text{Hom}_{\mathbf{Z}}(H, \mathcal{L}_{F_n}(k+1))$$

where $H := \text{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. (See Subsection 4.1 for details.) So far, we concentrate on the study of the cokernels of Johnson homomorphisms of $\text{Aut } F_n$ in a series of our works [25], [26], [28] and [5] with combinatorial group theory and representation theory. Since each of the Johnson homomorphisms is $\text{GL}(n, \mathbf{Z})$ -equivariant injective, it is an important problem to determine its images and cokernels. On the other hand, the study of the extendability of the Johnson homomorphisms have been received attentions. Morita [22] showed that the first Johnson homomorphism of the mapping class group, which initial domain is the Torelli group, can be extended to the mapping class group as a crossed homomorphism by using the extension theory of groups. Inspired by Morita's work, Kawazumi [17] obtained a corresponding results for $\text{Aut } F_n$ by using the Magnus expansion of F_n . Furthermore, he constructed higher twisted cohomology classes with the extended first Johnson homomorphism and the cup product. By restricting them to the mapping class group, he investigated relations between the higher cocycles and the Morita-Mumford classes. Recently, Day [4] showed that each of Johnson homomorphisms of $\text{Aut } F_n$ can be extended to a crossed homomorphism from $\text{Aut } F_n$ into a certain finitely generated free abelian group.

As mentioned above, we [9] constructed analogs of the Johnson homomorphisms with the algebra $\mathfrak{X}_{\mathbf{Q}}^2(F_n)$ of Fricke characters, and showed that the first homomorphism can be extended to $\text{Aut } F_n$ in [30]. It is, however, difficult to push forward with our research since the structures of the graded quotients $\text{gr}^k(\mathfrak{C})$ are too complicated to handle. In [31], we considered a similar situation for the $\text{SL}(2, \mathbf{C})$ -representation algebra $\mathfrak{R}_{\mathbf{Q}}^2(F_n)$. In this paper, we generalize our previous works to the $\text{SL}(m, \mathbf{C})$ -representation case. Set $s_{ij}(x) := a_{ij}(x) - \delta_{ij}$ for any $1 \leq i, j \leq m$ and $x \in F_n$ where δ means Kronecker's delta. Let \mathfrak{J}_{F_n} be the ideal of $\mathfrak{R}_{\mathbf{Q}}^m(F_n)$ generated by $s_{ij}(x_l)$ for any $1 \leq i, j \leq m$ and $1 \leq l \leq n$. Then the products of \mathfrak{J}_{F_n} define a descending filtration of $\mathfrak{R}_{\mathbf{Q}}^m(F_n)$: $\mathfrak{J}_{F_n} \supset \mathfrak{J}_{F_n}^2 \supset \mathfrak{J}_{F_n}^3 \supset \cdots$, which consists of $\text{Aut } F_n$ -invariant ideals. Set $\text{gr}^k(\mathfrak{J}_{F_n}) := \mathfrak{J}_{F_n}^k / \mathfrak{J}_{F_n}^{k+1}$

for any $k \geq 1$. Set

$$T_k := \left\{ \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \mid e_{ij, l} \geq 0, \sum_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \sum_{l=1}^n e_{ij, l} = k \right\} \subset \mathfrak{J}_{F_n}^k.$$

Theorem 1 (Propositions 6.1 and 6.5). *For each $k \geq 1$, the set $T_k \pmod{\mathfrak{J}_{F_n}^{k+1}}$ forms a basis of $\text{gr}^k(\mathfrak{J}_{F_n})$ as a \mathbf{Q} -vector space. Furthermore, for any $n \geq 2$ and $k \geq 1$, we have*

$$\text{gr}^k(\mathfrak{J}_{F_n}) \cong \bigoplus'_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \bigotimes S^{e_{ij}} H_{\mathbf{Q}}$$

as a $\text{GL}(n, \mathbf{Z})$ -module. Here the sum runs over all tuples (e_{ij}) for $1 \leq i, j \leq m$ and $(i, j) \neq (m, m)$ such that the sum of the e_{ij} is equal to k .

This theorem is a generalization of our previous result for the case where $k = 2$ in [31].

Now, set

$$\mathcal{D}_{F_n}^m(k) := \text{Ker}(\text{Aut } F_n \rightarrow \text{Aut}(\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^{k+1})).$$

The groups $\mathcal{D}_{F_n}^m(k)$ define a descending central filtration of $\text{Aut } F_n$. Let $\mathcal{A}_{F_n}(1) \supset \mathcal{A}_{F_n}(2) \supset \dots$ be the Andreadakis-Johnson filtration of $\text{Aut } F_n$. We show a relation between $\mathcal{A}_{F_n}(k)$ and $\mathcal{D}_{F_n}^m(k)$, and among $\mathcal{D}_{F_n}^m(k)$ s as follows.

Theorem 2 (Theorems 4.9 and 4.10).

- (1) For any $k \geq 1$, $\mathcal{A}_{F_n}(k) \subset \mathcal{D}_{F_n}^m(k)$.
- (2) For any $k \geq 1$ and $m \geq 2$, we have $\mathcal{D}_{F_n}^{m+1}(k) \subset \mathcal{D}_{F_n}^m(k)$.

In [31], we showed that $\mathcal{D}_{F_n}^2(k) = \mathcal{A}_{F_n}(k)$ for $1 \leq k \leq 4$. Hence, we have $\mathcal{D}_{F_n}^m(k) = \mathcal{A}_{F_n}(k)$ for any $1 \leq k \leq 4$. By referring to the theory of the Johnson homomorphisms, we can construct analogs of them:

$$\tilde{\eta}_k : \mathcal{D}_{F_n}^m(k) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^{k+1}(\mathfrak{J}_{F_n})).$$

defined by the corresponding $f \mapsto f^\sigma - f$ for any $f \in \mathfrak{J}_{F_n}$. (See Subsection 4.2 for details.) In this paper, we consider an extension of the first homomorphism $\tilde{\eta}_1$, and study some relations to the extension of $\tilde{\eta}_1$.

Set $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$. In [24], we computed $H^1(\text{Aut } F_n, H_{\mathbf{Q}}) = \mathbf{Q}$, and showed that it is generated by Morita's cocycle f_M . On the other hand, we [27] also computed $H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}) = \mathbf{Q}^{\oplus 2}$, and showed that it is generated by Kawazumi's cocycle f_K and the cocycle induced from f_M . Kawazumi's cocycle f_K is an extension of $\tilde{\eta}_1$. (See Subsection 6.2 for details.) In Subsection 6.2, we construct the crossed homomorphism

$$\theta_{F_n} : \text{Aut } F_n \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n}))$$

which is an extension of $\tilde{\eta}_1$. By taking suitable reductions of the target of θ_{F_n} , we obtain the crossed homomorphisms

$$f_1 : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}, \quad f_2 : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}.$$

Then we show the following.

Theorem 3 (Theorem 6.6). *For any $n \geq 2$,*

$$f_K = f_1, \quad f_M = -f_2 + \delta_x$$

for $x = x_1 + x_2 + \cdots + x_n \in H_{\mathbf{Q}}$ as crossed homomorphisms.

This shows that our crossed homomorphism induces both of Kawazumi's cocycle and Morita's cocycle, and that θ_{F_n} defines the non-trivial cohomology class in $H^1(\text{Aut } F_n, \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n})))$.

In [10] and [32], we studied $\mathfrak{X}_{\mathbf{Q}}^2(H)$ and $\mathfrak{R}_{\mathbf{Q}}^2(H)$. In this paper, we generalize the results in [32] to the $\text{SL}(m, \mathbf{C})$ -representation cases. By using a parallel argument, we can define a descending filtration $\mathfrak{J}_H \supset \mathfrak{J}_H^2 \supset \mathfrak{J}_H^3 \supset \cdots$ of ideals in $\mathfrak{R}_{\mathbf{Q}}^2(H)$. In contrast with the free group case, however, it is quite hard to determine the structure of the graded quotients $\text{gr}^k(\mathfrak{J}_H)$. Here, we gave basis of $\text{gr}^k(\mathfrak{J}_H)$ for $1 \leq k \leq 2$. In particular, we see

$$\text{gr}^1(\mathfrak{J}_H) \cong H_{\mathbf{Q}}^{\oplus m^2-1}, \quad \text{gr}^2(\mathfrak{J}_H) \cong (S^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}m^2(m^2-1)} \oplus (\Lambda^2 H_{\mathbf{Q}})^{\{\oplus \frac{1}{2}(m^2-1)(m^2-4)\}}.$$

We remark that for a general $m \geq 3$, the situation of the $\text{SL}(m, \mathbf{C})$ -representation case is much more different and complicated than those of the $\text{SL}(2, \mathbf{C})$ -representation case. At the present stage, we have no idea to give a result for a general $k \geq 3$. By using the above results, we construct the crossed homomorphism

$$\theta_H : \text{Aut } F_n \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \text{gr}^2(\mathfrak{J}_H)),$$

and show that it induces Morita's cocycle f_M in Theorem 7.4.

In Section 3, we study the $\text{SL}(m, \mathbf{C})$ -representation algebra for any group G , and introduce the descending central filtration $\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \cdots$ of $\text{Aut } G$. In Section 5, we discuss a general construction of a crossed homomorphism of $\text{Aut } G$ by using an associative algebra on which $\text{Aut } G$ acts. Then, in Sections 6 and 7, we apply the above results for the free group case and the free abelian group case respectively.

CONTENTS

1. Introduction	1
2. Notation and conventions	5
3. The $\text{SL}(m, \mathbf{C})$ -representation algebras of groups	5
4. Descending filtrations of $\text{Aut } G$	8
4.1. The Andreadakis-Johnson filtration	8
4.2. The filtration $\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \cdots$	9
5. Extensions of homomorphisms	12
5.1. Construction of crossed homomorphisms	12
5.2. Extensions of $\tilde{\eta}_1$	15
6. On the $\text{SL}(m, \mathbf{C})$ -representation algebra of F_n	16
6.1. The graded quotients $\text{gr}^k(\mathfrak{J}_{F_n})$	16
6.2. On the extension of $\tilde{\eta}_1$	19
7. On the $\text{SL}(m, \mathbf{C})$ -representation algebra of H	22
7.1. The graded quotients $\text{gr}^1(\mathfrak{J}_H)$	23
7.2. The graded quotients $\text{gr}^2(\mathfrak{J}_H)$	23
7.3. The crossed homomorphism θ_H	29

2. NOTATION AND CONVENTIONS

Throughout the paper, we use the following notation and conventions.

- Let G be a group. The automorphism group $\text{Aut } G$ of G acts on G from the right. For any $\sigma \in \text{Aut } G$ and $x \in G$, the action of σ on x is denoted by x^σ .
- Let N be a normal subgroup of a group G . For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion. Similarly, for an algebra R , an element $f \in R$ and an ideal I of R , we also denote by f the coset class of f in R/I if there is no confusion.
- For elements x and y in G , the commutator bracket $[x, y]$ of x and y is defined to be $xyx^{-1}y^{-1}$.
- On the direct product $\mathbf{N} \times \mathbf{N}$, we consider the usual lexicographic ordering among tuples (i, j) for $i, j \in \mathbf{N}$. The ordering is denoted by \leq_{lex} .

3. THE $\text{SL}(m, \mathbf{C})$ -REPRESENTATION ALGEBRAS OF GROUPS

Let G be a group generated by x_1, \dots, x_n . For any $m \geq 2$, we denote by $R^m(G)$ the set $\text{Hom}(G, \text{SL}(m, \mathbf{C}))$ of all $\text{SL}(m, \mathbf{C})$ -representations of G . Let $\mathcal{F}(R^m(G), \mathbf{C})$ be the set $\{\chi : R^m(G) \rightarrow \mathbf{C}\}$ of all complex-valued functions on $R^m(G)$. Then $\mathcal{F}(R^m(G), \mathbf{C})$ has the \mathbf{C} -algebra structure by the pointwise sum and product defined by

$$\begin{aligned} (\chi + \chi')(\rho) &:= \chi(\rho) + \chi'(\rho), \\ (\chi\chi')(\rho) &:= \chi(\rho)\chi'(\rho), \\ (\lambda\chi)(\rho) &:= \lambda(\chi(\rho)) \end{aligned}$$

for any $\chi, \chi' \in \mathcal{F}(R^m(G), \mathbf{C})$, $\lambda \in \mathbf{C}$, and $\rho \in R^m(G)$. The automorphism group $\text{Aut } G$ of G naturally acts on $R^m(G)$ and $\mathcal{F}(R^m(G), \mathbf{C})$ from the right by

$$\rho^\sigma(x) := \rho(x^{\sigma^{-1}}), \quad \rho \in R^m(G) \text{ and } x \in G$$

and

$$\chi^\sigma(\rho) := \chi(\rho^{\sigma^{-1}}), \quad \chi \in \mathcal{F}(R^m(G), \mathbf{C}) \text{ and } \rho \in R^m(G)$$

for any $\sigma \in \text{Aut } G$.

For any $x \in G$ and any $1 \leq i, j \leq m$, we define an element $a_{ij}(x)$ of $\mathcal{F}(R^m(G), \mathbf{C})$ to be

$$(a_{ij}(x))(\rho) := (i, j)\text{-component of } \rho(x)$$

for any $\rho \in R^m(G)$. The action of an element $\sigma \in \text{Aut } G$ on $a_{ij}(x)$ is given by $a_{ij}(x^\sigma)$. We have the following relations:

$$(1) \quad a_{ij}(x^{-1}) = \tilde{a}_{ji}(x)$$

and

$$(2) \quad a_{ij}(xy) = \sum_{k=1}^m a_{ik}(x)a_{kj}(y)$$

for any $1 \leq i, j \leq m$ and $x, y \in G$. Here $\tilde{a}_{ij}(x)$ denotes the (i, j) -cofactor of the matrix $(a_{ij}(x))$, and is written as a polynomial of $a_{kl}(x)$ s since the determinant of $(a_{ij}(x))$ is equal to one.

Let $\mathfrak{R}_{\mathbf{Q}}^m(G)$ be the \mathbf{Q} -subalgebra of $\mathcal{F}(R^m(G), \mathbf{C})$ generated by all $a_{ij}(x)$ for $x \in G$ and $1 \leq i, j \leq m$. We call $\mathfrak{R}_{\mathbf{Q}}^m(G)$ the $\mathrm{SL}(m, \mathbf{C})$ -representation algebras of G over \mathbf{Q} .

Lemma 3.1. *The algebra $\mathfrak{R}_{\mathbf{Q}}^m(G)$ is finitely generated by $a_{ij}(x_l)$ for $1 \leq l \leq n$.*

Proof. For any $x \in G$, we have $x = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_r}^{e_r}$ for some $1 \leq i_m \leq n$ and $e_j = \pm 1$. We show $a_{ij}(x) \in \mathfrak{R}_{\mathbf{Q}}^m(G)$ by the induction on $r \geq 1$. For $r = 1$, it is obvious that $a_{ij}(x) = a_{ij}(x_{i_1}) \in \mathfrak{R}_{\mathbf{Q}}^m(G)$ for $e_1 = 1$, and that $a_{ij}(x^{-1}) = \tilde{a}_{ji}(x_{i_1}) \in \mathfrak{R}_{\mathbf{Q}}^m(G)$ for $e_1 = -1$ from (1). For $r \geq 2$, from (2) we have

$$a_{ij}(x) = \sum_{k=1}^m a_{ik}(x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_{r-1}}^{e_{r-1}}) a_{kj}(x_{i_r}^{e_r})$$

Hence by the inductive hypothesis we see $a_{ij}(x) \in \mathfrak{R}_{\mathbf{Q}}^m(G)$. \square

Next, we consider an $\mathrm{Aut} G$ -invariant ideal of $\mathfrak{R}_{\mathbf{Q}}^m(G)$. Set $s_{ij}(x) := a_{ij}(x) - \delta_{ij}$ for any $1 \leq i, j \leq m$ and $x \in G$ where δ means the Kronecker's delta. Let \mathfrak{J}_G be the ideal of $\mathfrak{R}_{\mathbf{Q}}^m(G)$ generated by $s_{ij}(x_l)$ for any $1 \leq i, j \leq m$ and $1 \leq l \leq n$.

Lemma 3.2. *The ideal \mathfrak{J}_G is $\mathrm{Aut} G$ -invariant.*

Proof. Since $s_{ij}(x_l)^\sigma = s_{ij}(x_l^\sigma)$ for any $\sigma \in \mathrm{Aut} G$, it suffices to show that $s_{ij}(x) \in \mathfrak{J}_G$ for any $x \in G$.

From Lemma 3.1, $s_{ij}(x)$ is written as a polynomial among $a_{ij}(x_l)$. Then by substituting $s_{ij}(x) + \delta_{ij}$ for $a_{ij}(x)$, we see that

$$s_{ij}(x) = C + (\text{terms of degree} \geq 1)$$

for some $C \in \mathbf{Q}$. Let $\mathbf{1} : G \rightarrow \mathrm{SL}(m, \mathbf{C})$ be the trivial representation. By observing the image of $\mathbf{1}$ by $s_{ij}(x)$, we obtain $C = 0$ since $s_{ij}(x_l)(\mathbf{1}) = 0$ for any $1 \leq l \leq n$. This means $s_{ij}(x) \in \mathfrak{J}_G$. Therefore, \mathfrak{J}_G is $\mathrm{Aut} G$ -invariant. \square

We have a descending filtration

$$\mathfrak{J}_G \supset \mathfrak{J}_G^2 \supset \mathfrak{J}_G^3 \supset \cdots$$

which consists of $\mathrm{Aut} G$ -invariant ideals. Set $\mathrm{gr}^k(\mathfrak{J}_G) := \mathfrak{J}_G^k / \mathfrak{J}_G^{k+1}$ for any $k \geq 1$. Each graded quotient $\mathrm{gr}^k(\mathfrak{J}_G)$ is an $\mathrm{Aut} G$ -invariant finite dimensional \mathbf{Q} -vector space. Furthermore, the graded sum

$$\mathrm{gr}(\mathfrak{J}_G) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathfrak{J}_G)$$

naturally has the graded \mathbf{Q} -algebra structure given by the product map

$$\mathrm{gr}^k(\mathfrak{J}_G) \times \mathrm{gr}^l(\mathfrak{J}_G) \rightarrow \mathrm{gr}^{k+l}(\mathfrak{J}_G)$$

defined by

$$(f \pmod{\mathfrak{J}_G^{k+1}}, g \pmod{\mathfrak{J}_G^{l+1}}) \mapsto fg \pmod{\mathfrak{J}_G^{k+l+1}}.$$

Recall that

$$\begin{aligned}
s_{ij}(xy) &= s_{ij}(x) + s_{ij}(y) + \sum_{k=1}^m s_{ik}(x)s_{kj}(y) \\
&\equiv s_{ij}(x) + s_{ij}(y) \pmod{\mathfrak{J}_G^2}, \\
s_{ij}(x^{-1}) &= -s_{ij}(x) + \sum_{k=1}^m s_{ik}(x)s_{kj}(x^{-1}) \\
&\equiv -s_{ij}(x) \pmod{\mathfrak{J}_G^2}
\end{aligned}
\tag{3}$$

For any $x, y \in G$.

In order to describe generators of $\text{gr}^k(\mathfrak{J}_G)$ as a \mathbf{Q} -vector space, we consider relations among $a_{ij}(x_l)$ s. Set

$$\Delta_l := \det(a_{ij}(x_l)) = \sum_{\sigma \in \mathfrak{S}_m} a_{1\sigma(1)}(x_l) a_{2\sigma(2)}(x_l) \cdots a_{m\sigma(m)}(x_l) \in \mathfrak{R}_{\mathbf{Q}}^m(G)
\tag{4}$$

for any $1 \leq l \leq n$. Since we consider representations into $\text{SL}(m, \mathbf{C})$, we see that $\Delta_l = 1$ for any $1 \leq l \leq n$. We see

$$\begin{aligned}
\Delta_l - 1 &= -1 + (s_{11}(x_l) + 1)(s_{22}(x_l) + 1) \cdots (s_{mm}(x_l) + 1) \\
&\quad + \sum_{\substack{\sigma \in \mathfrak{S}_m \\ \sigma \neq 1}} (s_{1\sigma(1)}(x_l) + \delta_{1\sigma(1)}) \cdots (s_{m\sigma(m)}(x_l) + \delta_{m\sigma(m)}) \\
&= s_{11}(x_l) + s_{22}(x_l) + \cdots + s_{mm}(x_l) + (\text{terms of degree} \geq 2) \in \mathfrak{J}_G,
\end{aligned}$$

and hence

$$s_{11}(x_l) + s_{22}(x_l) + \cdots + s_{mm}(x_l) \equiv 0 \pmod{\mathfrak{J}_G^2}
\tag{5}$$

We give a generating set of $\text{gr}^k(\mathfrak{J}_G)$ as a \mathbf{Q} -vector space. For any $k \geq 1$, set

$$T_k := \left\{ \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \mid e_{ij, l} \geq 0, \sum_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \sum_{l=1}^n e_{ij, l} = k \right\} \subset \mathfrak{J}_G^k.$$

Lemma 3.3. *For each $k \geq 1$, T_k generates $\text{gr}^k(\mathfrak{J}_G)$ as a \mathbf{Q} -vector space.*

Proof. Since $\text{gr}^k(\mathfrak{J}_G)$ is generated by

$$\prod_{1 \leq i, j \leq m} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \text{ for } \sum_{1 \leq i, j \leq m} \sum_{l=1}^n e_{ij, l} = k,$$

by using (5) we see that T_k generates $\text{gr}^k(\mathfrak{J}_G)$. \square

Lemma 3.4. *Let $\varphi : G \rightarrow G'$ be a group homomorphism. Then φ induces the algebra homomorphism*

$$\overline{\varphi} : \mathfrak{R}_{\mathbf{Q}}^m(G) \rightarrow \mathfrak{R}_{\mathbf{Q}}^m(G')$$

defined by

$$a_{ij}(x) \mapsto a_{ij}(\varphi(x))$$

for any $x \in G$. Furthermore, if φ is surjective, so is $\overline{\varphi}$.

Proof. For a given homomorphism $\varphi : G \rightarrow G'$, φ induces the map $\tilde{\varphi} : R^m(G') \rightarrow R^m(G)$ defined by $\rho \mapsto \rho \circ \varphi$. Then $\tilde{\varphi}$ induces the algebra homomorphism $\overline{\varphi} : \mathcal{F}(R^m(G), \mathbf{C}) \rightarrow \mathcal{F}(R^m(G'), \mathbf{C})$ defined by

$$\overline{\varphi}(\chi) := \chi \circ \tilde{\varphi}$$

for any $\chi \in R^m(G')$. The restriction of $\overline{\varphi}$ to $\mathfrak{R}_{\mathbf{Q}}^m(G)$ is the required homomorphism. \square

4. DESCENDING FILTRATIONS OF $\text{Aut } G$

4.1. The Andreadakis-Johnson filtration.

In this subsection, we review the Andreadakis-Johnson filtration of $\text{Aut } G$ without proofs. The main purpose of the subsection is to fix the notations. For basic materials concerning the Andreadakis-Johnson filtration and the Johnson homomorphisms of $\text{Aut } G$, see [26] or [29], for example.

First, consider the lower central series $\Gamma_G(1) \supset \Gamma_G(2) \supset \cdots$ of G defined by the rule

$$\Gamma_G(1) := G, \quad \Gamma_G(k) := [\Gamma_G(k-1), G], \quad k \geq 2.$$

For any $y_1, \dots, y_k \in G$, the left-normed commutator

$$[[\cdots [[y_1, y_2], y_3], \cdots], y_k]$$

of weight k is denoted by

$$[y_1, y_2, \cdots, y_k]$$

for simplicity. Then we have

Lemma 4.1 (See Section 5.3 in [20].). *For any $k \geq 1$, the group $\Gamma_G(k)$ is generated by all left-normed commutators of weight k .*

For any $k \geq 1$, set $\mathcal{L}_G(k) := \Gamma_G(k)/\Gamma_G(k+1)$.

Lemma 4.2. *If G is generated by x_1, \dots, x_n , then each of the graded quotients $\mathcal{L}_G(k)$ is generated by (the coset class of) the simple k -fold commutators*

$$[x_{i_1}, x_{i_2}, \dots, x_{i_k}], \quad 1 \leq i_j \leq n$$

as an abelian group.

For a proof, see Theorem 5.4 in [20], for example. This shows that if G is finitely generated then so is $\mathcal{L}_G(k)$ for any $k \geq 1$.

For $k \geq 1$, the action of $\text{Aut } G$ on each nilpotent quotient $G/\Gamma_G(k+1)$ induces the homomorphism

$$\text{Aut } G \rightarrow \text{Aut}(G/\Gamma_G(k+1)).$$

We denote its kernel by $\mathcal{A}_G(k)$. Then the groups $\mathcal{A}_G(k)$ define a descending filtration

$$\mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \cdots \supset \mathcal{A}_G(k) \supset \cdots$$

of $\text{Aut } G$. We call this the Andreadakis-Johnson filtration of $\text{Aut } G$. The first term $\mathcal{A}_G(1)$ is called the IA-automorphism group of G , and is also denoted by $\text{IA}(G)$. Namely, $\text{IA}(G)$ consists of automorphisms which act on the abelianization G^{ab} of G trivially.

The Andreadakis-Johnson filtration of $\text{Aut } G$ was originally introduced by Andreadakis [2] in the 1960s. The name “Johnson” comes from Dennis Johnson who studied the Johnson filtration and the Johnson homomorphism for the mapping class group of a surface in the 1980s. In particular, Andreadakis showed that

Theorem 4.3 (Andreadakis, [2]).

- (1) For any $k, l \geq 1$, $\sigma \in \mathcal{A}_G(k)$ and $x \in \Gamma_G(l)$, $x^{-1}x^\sigma \in \Gamma_G(k+l)$.
- (2) For any k and $l \geq 1$, $[\mathcal{A}_G(k), \mathcal{A}_G(l)] \subset \mathcal{A}_G(k+l)$.
- (3) If $\bigcap_{k \geq 1} \Gamma_G(k) = 1$, then $\bigcap_{k \geq 1} \mathcal{A}_G(k) = 1$.

To study the structure of the graded quotients $\text{gr}^k(\mathcal{A}_G) := \mathcal{A}_G(k)/\mathcal{A}_G(k+l)$, we use the Johnson homomorphisms of $\text{Aut } G$. For any $\sigma \in \mathcal{A}_G(k)$, consider a map $\tilde{\tau}_k(\sigma) : G^{\text{ab}} \rightarrow \mathcal{L}_G(k+1)$ defined by

$$x \pmod{\Gamma_G(2)} \mapsto x^{-1}x^\sigma \pmod{\Gamma_G(k+2)}$$

for any $x \in G$. Then $\tilde{\tau}_k(\sigma)$ is a homomorphism between abelian groups. Furthermore, a map $\tilde{\tau}_k : \mathcal{A}_G(k) \rightarrow \text{Hom}_{\mathbf{Z}}(G^{\text{ab}}, \mathcal{L}_G(k+1))$ defined by

$$\sigma \mapsto \tilde{\tau}_k(\sigma)$$

is a homomorphism. From the definition, it is easy to see that the kernel of $\tilde{\tau}_k$ is $\mathcal{A}_G(k+1)$. Therefore $\tilde{\tau}_k$ induces the injective homomorphism

$$\tau_k : \text{gr}^k(\mathcal{A}_G) \hookrightarrow \text{Hom}_{\mathbf{Z}}(G^{\text{ab}}, \mathcal{L}_G(k+1)).$$

For each $k \geq 1$, we call both of the homomorphisms $\tilde{\tau}_k$ and τ_k the k -th Johnson homomorphisms of $\text{Aut } G$.

4.2. The filtration $\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \dots$.

For $k \geq 1$, the action of $\text{Aut } G$ on each nilpotent quotient $\mathfrak{J}_G/\mathfrak{J}_G^{k+1}$ induces the homomorphism $\text{Aut } G \rightarrow \text{Aut}(\mathfrak{J}_G/\mathfrak{J}_G^{k+1})$. Set

$$\mathcal{D}_G^m(k) := \text{Ker}(\text{Aut } G \rightarrow \text{Aut}(\mathfrak{J}_G/\mathfrak{J}_G^{k+1})).$$

The groups $\mathcal{D}_G^m(k)$ define the descending filtration

$$\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \dots \supset \mathcal{D}_G^m(k) \supset \dots$$

of $\text{Aut } G$. For any $f \in \mathfrak{J}_G$ and $\sigma \in \text{Aut } G$, set

$$s_\sigma(f) := f^\sigma - f \in \mathfrak{J}_G.$$

In the following, we give a few lemmas and proposition without proofs. In [31], we give proofs of the lemmas and propositions for the case where $m = 2$ and $G = F_n$. We show the followings by the same way as in [31]. For details, see [31].

Lemma 4.4. For any $f \in \mathfrak{J}_G$ and $\sigma, \tau \in \text{Aut } G$,

- (1) $s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f)$,
- (2) $s_{1_G}(f) = 0$,
- (3) $s_{\sigma^{-1}}(f) = -(s_\sigma(f))^{\sigma^{-1}}$,
- (4) $s_{[\sigma, \tau]}(f) = \{s_\tau(s_\sigma(f)) - s_\sigma(s_\tau(f))\}^{\sigma^{-1}\tau^{-1}}$.

Lemma 4.5. For any $k, l \geq 1$, $f \in \mathfrak{J}_G^l$ and $\sigma \in \mathcal{D}_G^m(k)$, we have $s_\sigma(f) \in \mathfrak{J}_G^{k+l}$.

Proposition 4.6. For any $k, l \geq 1$, $[\mathcal{D}_G^m(k), \mathcal{D}_G^m(l)] \subset \mathcal{D}_G^m(k+l)$.

This proposition shows that the filtration $\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \dots$ is a central filtration of $\text{Aut } G$. Next, we show that $\mathcal{D}_G^m(k)$ contains $\mathcal{A}_G(k)$ for any $k \geq 1$.

Lemma 4.7. *For any $1 \leq i, j \leq m$, $k \geq 1$ and $y \in \Gamma_G(k)$, we have $s_{ij}(y) \in \mathfrak{J}_G^k$.*

Proof. From the fact that $\Gamma_G(k)$ is generated by all left-normed commutators, and from (3), it suffices to show the lemma in the case where $y = [y_1, y_2, \dots, y_k]$ for any $y_1, \dots, y_k \in F_n$. We use the induction on $k \geq 1$. If $k = 1$, the lemma is obvious. Assume $k \geq 2$ and set $z := [y_1, \dots, y_{k-1}]$. Then $s_{ij}(z) \in \mathfrak{J}_G^{k-1}$ by the inductive hypothesis. Furthermore, from (1), we see $s_{ij}(z^{-1}) \in \mathfrak{J}_G^{k-1}$. By using (2), we have

$$\begin{aligned} s_{ij}([z, y_k]) &= s_{ij}(zy_k z^{-1} y_k^{-1}) \\ &= s_{ij}(zy_k) + s_{ij}(z^{-1} y_k^{-1}) + \sum_{h=1}^m s_{ih}(zy_k) s_{hj}(z^{-1} y_k^{-1}) \\ &\equiv s_{ij}(z) + s_{ij}(y_k) + s_{ij}(z^{-1}) + s_{ij}(y_k^{-1}) + \sum_{h=1}^m s_{ih}(y_k) s_{hj}(y_k^{-1}) \pmod{\mathfrak{J}_G^k} \\ &\equiv s_{ij}(z) + s_{ij}(z^{-1}) \pmod{\mathfrak{J}_G^k} \\ &\equiv 0 \pmod{J^k}. \end{aligned}$$

Here, the last equation follows from

$$0 = s_{ij}(1_G) = s_{ij}(zz^{-1}) \equiv s_{ij}(z) + s_{ij}(z^{-1}) \pmod{\mathfrak{J}_G^k}.$$

□

Lemma 4.8. *For any $k \geq 1$, $z \in G$ and $y \in \Gamma_G(k)$, we have $s_{ij}(zy) \equiv s_{ij}(y) \pmod{\mathfrak{J}_G^k}$.*

Proof. This lemma follows from (2) and Lemma 4.7 immediately. □

From Lemma 4.8, we obtain the following theorem.

Theorem 4.9. *For any $k \geq 1$, $\mathcal{A}_G(k) \subset \mathcal{D}_G^m(k)$.*

Now, we give a relation between $\mathcal{D}_G^m(k)$ and $\mathcal{D}_G^{m+1}(k)$ for $m \geq 2$.

Theorem 4.10. *For any $k \geq 1$ and $m \geq 2$, we have $\mathcal{D}_G^{m+1}(k) \subset \mathcal{D}_G^m(k)$.*

Proof. Take any $\sigma \in \mathcal{D}_G^{m+1}(k)$ and $x \in G$. For any $1 \leq i, j \leq m$, in order to distinguish $s_{ij}(x) \in \mathfrak{R}_{\mathbf{Q}}^{m+1}(G)$ from $s_{ij}(x) \in \mathfrak{R}_{\mathbf{Q}}^m(G)$, we write $s_{ij}^m(x)$ for $s_{ij}(x)$ if we consider $s_{ij}(x)$ as an element in $\mathfrak{R}_{\mathbf{Q}}^m(G)$. For any representation $\rho : G \rightarrow \mathrm{SL}(m, \mathbf{C})$, define the representation $\bar{\rho} : G \rightarrow \mathrm{SL}(m+1, \mathbf{C})$ by

$$\bar{\rho}(x) := \begin{pmatrix} \rho(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we see

$$\begin{aligned} s_{ij}^m(x^\sigma)(\rho) &= s_{ij}^m(\rho(x^\sigma)) = s_{ij}^{m+1}(\bar{\rho}(x^\sigma)) = s_{ij}^{m+1}(x^\sigma)(\bar{\rho}) \\ &= (s_{ij}^{m+1}(x) + (\text{polynomial among } s_{pq}^{m+1}(y) \text{ of degree } \geq k+1))(\bar{\rho}) \\ &= s_{ij}^{m+1}(\bar{\rho}(x)) + (\text{polynomial among } s_{pq}^{m+1}(\bar{\rho}(y)) \text{ of degree } \geq k+1) \\ &= s_{ij}^m(\rho(x)) + (\text{polynomial among } s_{pq}^m(\rho(y)) \text{ of degree } \geq k+1) \\ &= s_{ij}^m(x)(\rho) + (\text{polynomial among } s_{pq}^m(\rho(y)) \text{ of degree } \geq k+1). \end{aligned}$$

Hence we see $s_{ij}^m(x^\sigma) \equiv s_{ij}^m(x) \pmod{\mathfrak{J}_G^{k+1}}$ in $\mathfrak{R}_{\mathbf{Q}}^m(G)$. This shows $\sigma \in \mathcal{D}_G^m(k)$. □

From the above theorem, we have

$$\mathcal{A}_G(k) \subset \cdots \subset \mathcal{D}_G^{m+1}(k) \subset \mathcal{D}_G^m(k) \subset \cdots \subset \mathcal{D}_G^2(k).$$

Here we give a sufficient condition for $\mathcal{A}_G(k) = \mathcal{D}_G^m(k)$. For any $1 \leq i, j \leq m$ and $k \geq 1$, consider the homomorphism $s_{ij}^{(k)} : \Gamma_G(k) \rightarrow \text{gr}^k(\mathfrak{J}_G)$ defined by

$$x \mapsto s_{ij}(x).$$

By Lemma 4.7, $s_{ij}^{(k)}$ naturally induces the homomorphism $\mathcal{L}_G(k) \rightarrow \text{gr}^k(\mathfrak{J}_G)$, which is also denoted by $s_{ij}^{(k)}$ by abuse of language.

Proposition 4.11. *Assume $\bigcap_{k \geq 1} \Gamma_G(k) = \{1\}$. Let k be a positive integer. For any $1 \leq p \leq k$, assume*

$$\bigcap_{1 \leq i, j \leq m} \text{Ker}(s_{ij}^{(p)}) = \{0\}.$$

Then $\mathcal{D}_G^m(k) \subset \mathcal{A}_G(k)$.

Proof. Assume that there exists some $\sigma \in \mathcal{D}_G^m(k)$ such that $\sigma \notin \mathcal{A}_G(k)$. Since $\mathcal{D}_G^m(k) \subset \mathcal{D}_G^m(1) = \mathcal{A}_G(1)$, there exists some $1 \leq p \leq k-1$ such that $\sigma \in \mathcal{A}_G(p) \setminus \mathcal{A}_G(p+1)$. Thus, we have some $x \in G$ such that $x^{-1}x^\sigma \in \Gamma_G(p+1)$ and $x^{-1}x^\sigma \notin \Gamma_G(p+2)$. By the assumption, for some $1 \leq i, j \leq m$, we see $s_{ij}(x^{-1}x^\sigma)$ does not belong to \mathfrak{J}_G^{m+2} . Set $\gamma := x^{-1}x^\sigma \in \Gamma_G(p+1)$, then

$$\begin{aligned} s_{ij}(x^\sigma) &= s_{ij}(x\gamma) \\ &= s_{ij}(x) + s_{ij}(\gamma) + \sum_{h=1}^m s_{ih}(x)s_{hj}(\gamma) \\ &\equiv s_{ij}(x) + s_{ij}(\gamma) \pmod{\mathfrak{J}_{F_n}^{p+2}}. \end{aligned}$$

On the other hand, since $\sigma \in \mathcal{D}_G^m(k)$, we have $s_{ij}(x^\sigma) - s_{ij}(x) \in \mathfrak{J}_G^{k+1} \subset \mathfrak{J}_G^{p+2}$. This is the contradiction. Therefore $\sigma \in \mathcal{A}_G(k)$. \square

Finally, we introduce Johnson homomorphism like homomorphisms to study the graded quotients $\text{gr}^k(\mathcal{D}_G^m) := \mathcal{D}_G^m(k)/\mathcal{D}_G^m(k+1)$. To begin with, we consider the action of $\text{Aut } G$. Since each $\mathcal{D}_G^m(k)$ is a normal subgroup of $\text{Aut } G$, the group $\text{Aut } G$ naturally acts on $\text{gr}^k(\mathcal{D}_G^m)$ by the conjugation from the right. Furthermore since $\mathcal{D}_G^m(1) \supset \mathcal{D}_G^m(2) \supset \cdots$ is a central filtration, the action of $\mathcal{D}_G^m(1)$ on $\text{gr}^k(\mathcal{D}_G^m)$ is trivial. Hence we can consider each $\text{gr}^k(\mathcal{D}_G^m)$ as an $\text{Aut } G/\mathcal{D}_G^m(1)$ -module.

For any $k \geq 1$ and $\sigma \in \mathcal{D}_G^m(k)$, define the map $\tilde{\eta}_k(\sigma) : \text{gr}^1(\mathfrak{J}_G) \rightarrow \text{gr}^{k+1}(\mathfrak{J}_G)$ to be

$$\tilde{\eta}_k(\sigma)(f) := s_\sigma(f) = f^\sigma - f \in \text{gr}^{k+1}(\mathfrak{J}_G)$$

for any $f \in \mathfrak{J}_G$. The well-definedness of the map $\tilde{\eta}_k(\sigma)$ follows from Lemma 4.5. It is easily seen that $\tilde{\eta}_k(\sigma)$ is a homomorphism. Then we have the map $\tilde{\eta}_k : \mathcal{D}_G^m(k) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^{k+1}(\mathfrak{J}_G))$ defined by $\sigma \mapsto \tilde{\eta}_k(\sigma)$. For any $\sigma, \tau \in \mathcal{D}_G^m(k)$, from (1) of Lemma 4.4, and from Lemma 4.5, we see

$$s_{\sigma\tau}(f) = (s_\sigma(f))^\tau + s_\tau(f) \equiv s_\sigma(f) + s_\tau(f) \pmod{\mathfrak{J}_G^{k+2}}.$$

This shows that $\tilde{\eta}_k$ is a homomorphism. By the definition, the kernel of $\tilde{\eta}_k$ is $\mathcal{D}_G^m(k+1)$. Thus $\tilde{\eta}_k$ induces the injective homomorphism

$$\eta_k : \text{gr}^k(\mathcal{D}_G^m) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^{k+1}(\mathfrak{J}_G)).$$

The homomorphism η_k is an $\text{Aut } G/\mathcal{D}_G^m(1)$ -equivariant homomorphism. By using the homomorphisms η_k , we can consider $\text{gr}^k(\mathcal{D}_G^m)$ as a submodule of $\text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^{k+1}(\mathfrak{J}_G))$, and hence we obtain

- (1) Each of $\text{gr}^k(\mathcal{D}_G^m)$ is torsion-free.
- (2) $\dim_{\mathbf{Q}}(\text{gr}^k(\mathcal{D}_G^m) \otimes_{\mathbf{Z}} \mathbf{Q}) < \infty$.

If $\mathcal{D}_G^m(k) = \mathcal{A}_G(k)$, then the above facts immediately follows from Andreadakis's result for the Andreadakis-Johnson filtration in [2].

5. EXTENSIONS OF HOMOMORPHISMS

In this section, we consider to extend the homomorphisms η_k and η'_k to $\text{Aut } G$ as crossed homomorphisms by using an extension of associative algebras. For basic materials for cohomology of associative algebras, see Chapter IX in [3]. For an associative algebra \mathcal{R} , we denote by $\text{Aut}_{(\text{Alg})}(\mathcal{R})$ the algebra automorphism group of \mathcal{R} . For a \mathbf{Q} -vector space M , we denote by $\text{Aut } M$ the \mathbf{Q} -linear automorphism group of M .

5.1. Construction of crossed homomorphisms.

Let \mathcal{R} be a \mathbf{Q} -algebra generated by t_1, \dots, t_n , and \mathcal{J} the ideal of \mathcal{R} generated by t_1, \dots, t_n . Then we have a descending filtration $\mathcal{J} \supset \mathcal{J}^2 \supset \dots$ of \mathcal{R} . First, consider the extension

$$(6) \quad 0 \rightarrow \mathcal{J}^2/\mathcal{J}^3 \rightarrow \mathcal{R}/\mathcal{J}^3 \rightarrow \mathcal{R}/\mathcal{J}^2 \rightarrow 0$$

of associative \mathbf{Q} -algebras.

Proposition 5.1. *The natural homomorphism*

$$\Phi : \text{Aut}_{(\text{Alg})}(\mathcal{R}/\mathcal{J}^3) \rightarrow \text{Aut}_{(\text{Alg})}(\mathcal{R}/\mathcal{J}^2)$$

is surjective.

Proof. First, observe the characteristic class $\kappa(\text{id}_{\mathcal{J}^2/\mathcal{J}^3})$ of the extension (6) where

$$\begin{aligned} \kappa : \text{Hom}_{E(\mathcal{R}/\mathcal{J}^2)}(\mathcal{J}^2/\mathcal{J}^3, \mathcal{J}^2/\mathcal{J}^3) &\rightarrow \text{Ext}_{E(\mathcal{R}/\mathcal{J}^2)}^1(\mathcal{R}/\mathcal{J}^2, \mathcal{J}^2/\mathcal{J}^3) \\ &\rightarrow H^2(\mathcal{R}/\mathcal{J}^2, \mathcal{J}^2/\mathcal{J}^3) \end{aligned}$$

is the homomorphism induced from the connecting homomorphism, and $E(\mathcal{R}/\mathcal{J}^2)$ is the enveloping algebra of $\mathcal{R}/\mathcal{J}^2$. Choose a 2-cocycle c of the algebra $\mathcal{R}/\mathcal{J}^2$ with coefficients in $\mathcal{J}^2/\mathcal{J}^3$, which represents the cohomology class $\kappa(\text{id}_{\mathcal{J}^2/\mathcal{J}^3})$. Then $\mathcal{R}/\mathcal{J}^3$ can be explicitly described as the product $\mathcal{J}^2/\mathcal{J}^3 \times \mathcal{R}/\mathcal{J}^2$ equipped with the multiplication given by

$$(\xi, \tau)(\xi', \tau') = (\xi\tau' + \tau\xi' + c(\tau, \tau'), \tau\tau')$$

for any $(\xi, \tau), (\xi', \tau') \in \mathcal{J}^2/\mathcal{J}^3 \times \mathcal{R}/\mathcal{J}^2$. We denote by Λ_c this associative algebra.

For any $\alpha \in \text{Aut}_{(\text{Alg})}(\mathcal{R}/\mathcal{J}^2)$, since the cohomology class of c is the characteristic class induced from $\text{id}_{\mathcal{J}^2/\mathcal{J}^3}$, the 2-cocycle $\alpha^\sharp(c)$ should be cohomologous to c where α^\sharp is the induced homomorphism from α . Hence there exists a 1-chain $d : \mathcal{R}/\mathcal{J}^2 \rightarrow \mathcal{J}^2/\mathcal{J}^3$ such that $\alpha^\sharp(c) - c = \delta d$ where δ is the coboundary homomorphism. Namely, we have

$$\alpha^{-1} \cdot c(\alpha(\tau), \alpha(\tau')) - c(\tau, \tau') = \tau d(\tau') - d(\tau\tau') + d(\tau)\tau'$$

for any $\tau, \tau' \in \mathcal{R}/\mathcal{J}^2$. Define the map $\tilde{\alpha} : \Lambda_c \rightarrow \Lambda_c$ to be

$$(\xi, \tau) \mapsto (\alpha(\xi) - \alpha(d(\tau)), \alpha(\tau)).$$

Then we see that $\tilde{\alpha} \in \text{Aut}_{(\text{Alg})}(\mathcal{R}/\mathcal{J}^3)$ under the identification $\Lambda_c = \mathcal{R}/\mathcal{J}^3$, and $\Phi(\tilde{\alpha}) = \alpha$. \square

For any $f \in \mathcal{J}$, we denote the coset class of f in $\mathcal{J}/\mathcal{J}^k$ by $[f]_k$. For any $k \geq 2$, set

$$\overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^k) := \{\sigma \in \text{Aut}(\mathcal{J}/\mathcal{J}^k) \mid \sigma([\gamma\gamma']_k) = \sigma([\gamma]_k) \sigma([\gamma']_k), \gamma, \gamma' \in \mathcal{J}\}.$$

Note that $\overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2) = \text{Aut}(\mathcal{J}/\mathcal{J}^2) = \text{GL}_{\mathbf{Q}}(\mathcal{J}/\mathcal{J}^2)$.

Lemma 5.2. *The group homomorphism*

$$\Psi : \text{Aut}_{(\text{Alg})}(\mathcal{R}/\mathcal{J}^k) \rightarrow \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^k)$$

defined by $\sigma \mapsto \sigma|_{\mathcal{J}/\mathcal{J}^k}$ is an isomorphism.

Proof. Consider the polynomial algebra $\mathbf{Q}[s_1, \dots, s_n]$, and the surjection $\pi : \mathbf{Q}[s_1, \dots, s_n] \rightarrow \mathcal{R}$ given by $1 \mapsto 1$ and $s_i \mapsto t_i$. The kernel \mathcal{I} of π is contained in the ideal \mathcal{J}' generated by s_1, \dots, s_n .

We construct the inverse map of Ψ . For any $\beta \in \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^k)$, define the \mathbf{Q} -algebra homomorphism $\tilde{\beta} : \mathbf{Q}[s_1, \dots, s_n] \rightarrow \mathcal{R}/\mathcal{J}^k$ to be

$$\tilde{\beta}(1) := [1]_k, \quad \tilde{\beta}(s_i) := \beta([t_i]_k).$$

Since $\mathcal{I} \subset \text{Ker}(\tilde{\beta})$ and $\tilde{\beta}(\mathcal{J}') = \mathcal{J}$, the above $\tilde{\beta}$ induces the algebra homomorphism $\mathcal{R}/\mathcal{J}^k \rightarrow \mathcal{R}/\mathcal{J}^k$, say $\tilde{\beta}$ by the abuse of the notation. Since β is an automorphism, so is $\tilde{\beta}$. Then we have the homomorphism $\Psi' : \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^k) \rightarrow \text{Aut}(\mathcal{R}/\mathcal{J}^k)$ defined by $\beta \mapsto \tilde{\beta}$, and see that Ψ' is the inverse of Ψ . \square

From Proposition 5.1 and Lemma 5.2, we obtain the induced surjective homomorphism

$$\varphi : \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3) \rightarrow \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2).$$

Next, we consider an embedding of $\text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$ into $\overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$. For any $f \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$, define the map $\tilde{f} : \mathcal{J}/\mathcal{J}^3 \rightarrow \mathcal{J}/\mathcal{J}^3$ by

$$\tilde{f}([\gamma]_3) := [\gamma]_3 + f([\gamma]_2)$$

for any $\gamma \in \mathcal{J}$.

Proposition 5.3. *With the above notation, for any $f \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$, we see $\tilde{f} \in \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$, and the map*

$$\iota : \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J})) \rightarrow \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$$

defined by $f \mapsto \tilde{f}$ is injective.

Proof. First, we show \tilde{f} is a homomorphism. For any $\gamma, \gamma' \in \mathcal{J}$,

$$\begin{aligned} \tilde{f}([\gamma]_3 + [\gamma']_3) &= \tilde{f}([\gamma + \gamma']_3) = [\gamma + \gamma']_3 + f([\gamma + \gamma']_2) \\ &= ([\gamma]_3 + f([\gamma]_2)) + ([\gamma']_3 + f([\gamma']_2)) \\ &= \tilde{f}([\gamma]_3) + \tilde{f}([\gamma']_3). \end{aligned}$$

Thus \tilde{f} is a homomorphism. Furthermore, \tilde{f} satisfies

$$\begin{aligned} \tilde{f}([\gamma]_3[\gamma']_3) &= \tilde{f}([\gamma\gamma']_3) = [\gamma\gamma']_3 + f([\gamma\gamma']_2) = [\gamma]_3[\gamma']_3 \\ &= ([\gamma]_3 + f([\gamma]_2))([\gamma']_3 + f([\gamma']_2)) = \tilde{f}([\gamma]_3)\tilde{f}([\gamma']_3) \end{aligned}$$

for any $\gamma, \gamma' \in \mathcal{J}$. On the other hand, we have $\widetilde{f+g} = \widetilde{f} \circ \widetilde{g}$ for any $f, g \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$. In fact, for any $\gamma \in \mathcal{J}$,

$$\begin{aligned} (\widetilde{f+g})([\gamma]_3) &= [\gamma]_3 + f([\gamma]_2) + g([\gamma]_2), \\ (\widetilde{f} \circ \widetilde{g})([\gamma]_3) &= \widetilde{f}([\gamma]_3 + g([\gamma]_2)) = [\gamma]_3 + g([\gamma]_2) + f([\gamma]_2). \end{aligned}$$

This shows that ι is a homomorphism. On the other hand, for the zero map $0 \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$, it is obvious that $\widetilde{0} = \text{id}_{\mathcal{J}/\mathcal{J}^3}$. Hence each \widetilde{f} has its inverse map $\widetilde{f}^{-1} = \widetilde{-f}$. This means \widetilde{f} an automorphism on $\mathcal{J}/\mathcal{J}^3$.

Finally, we show that ι is injective. Assume that $\widetilde{f} = \text{id}_{\mathcal{J}/\mathcal{J}^3}$ for some $f \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$. Then for any $\gamma \in \mathcal{J}$, we have

$$\widetilde{f}([\gamma]_3) = [\gamma]_3 + f([\gamma]_2) = [\gamma]_3.$$

Hence we obtain $f([\gamma]_2) = 0$ for any $[\gamma]_2 \in \mathcal{J}/\mathcal{J}^2$, and $f = 0$. This shows ι is injective. \square

Proposition 5.4. *The sequence*

$$(7) \quad 0 \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J})) \xrightarrow{\iota} \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3) \xrightarrow{\varphi} \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2) \rightarrow 1$$

is a split group extension.

Proof. First, we show the above sequence is exact. Namely, it suffices to show $\text{Im}(\iota) = \text{Ker}(\varphi)$. The fact that $\text{Im}(\iota) \subset \text{Ker}(\varphi)$ follows from

$$\begin{aligned} (\varphi \circ \iota)(f)([\gamma]_2) &= \varphi(\widetilde{f})([\gamma]_2) = [\gamma]_3 + f([\gamma]_2) \pmod{\mathcal{J}^2} \\ &= [\gamma]_2 \end{aligned}$$

for any $f \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$ and $\gamma \in \mathcal{J}$. To show $\text{Im}(\iota) \supset \text{Ker}(\varphi)$, take any $f \in \text{Ker}(\varphi)$. Define $g \in \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J}))$ to be

$$(8) \quad g([\gamma]_2) := f([\gamma]_3) - [\gamma]_3$$

for any $\gamma \in \mathcal{J}$. The map g is well-defined. In fact, for any $\gamma, \gamma' \in \mathcal{J}$ such that $[\gamma]_2 = [\gamma']_2$, if we set $\gamma' - \gamma = \varepsilon \in \mathcal{J}^2$, then we have

$$\begin{aligned} g([\gamma']_2) &= f([\gamma']_3) - [\gamma']_3 = f([\gamma + \varepsilon]_3) - [\gamma + \varepsilon]_3 \\ &= f([\gamma]_3) - [\gamma]_3 + f([\varepsilon]_3) - [\varepsilon]_3 = f([\gamma]_3) - [\gamma]_3 \\ &= g([\gamma]_2). \end{aligned}$$

Here we remark that $f([\varepsilon]_3) - [\varepsilon]_3 = 0 \in \mathcal{J}^2/\mathcal{J}^3$ since $\varepsilon \in \mathcal{J}^2$ and $f \in \text{Ker}(\varphi)$. It is easy to show that g is a homomorphism. Furthermore, for any $\gamma \in \mathcal{J}$,

$$\widetilde{g}([\gamma]_3) = [\gamma]_3 + g([\gamma]_2) = f([\gamma]_3).$$

This shows $f = \widetilde{g} = \iota(g) \in \text{Im}(\iota)$.

Finally, we construct the section $s : \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2) \rightarrow \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$ of (7). Take elements $\gamma_1, \gamma_2, \dots, \gamma_p \in \mathcal{J}$ and $\gamma_{p+1}, \dots, \gamma_{p+q} \in \mathcal{J}^2$ such that $([\gamma_1]_2, [\gamma_2]_2, \dots, [\gamma_p]_2)$ and $([\gamma_{p+1}]_3, \dots, [\gamma_{p+q}]_3)$ form bases of $\text{gr}^1(\mathcal{J})$ and $\text{gr}^2(\mathcal{J})$ respectively. Then $([\gamma_1]_3, [\gamma_2]_3, \dots, [\gamma_{p+q}]_3)$ is a basis of $\mathcal{J}/\mathcal{J}^3$.

For any $\beta \in \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2)$, there exists an element $\tilde{\beta} \in \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$ such that $\varphi(\tilde{\beta}) = \beta$. In general, for any $1 \leq j \leq p$, the image $\tilde{\beta}([\gamma_j]_3)$ can be written as

$$\tilde{\beta}([\gamma_j]_3) = a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3$$

for some $a_{ij} \in \mathbf{Q}$. Since $\beta \in \text{Aut}(\mathcal{J}/\mathcal{J}^2)$, if we set

$$v_j := a_{1j}[\gamma_1]_2 + \cdots + a_{pj}[\gamma_p]_2$$

for any $1 \leq j \leq p$, then (v_1, v_2, \dots, v_p) is a basis of $\text{gr}^1(\mathcal{J})$. Let $\delta = \delta_{\tilde{\beta}} : \text{gr}^1(\mathcal{J}) \rightarrow \text{gr}^2(\mathcal{J})$ be the \mathbf{Q} -linear map given by

$$\delta(v_j) = -(a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3)$$

for any $1 \leq j \leq p$. Then we obtain

$$\begin{aligned} (\tilde{\delta} \circ \tilde{\beta})([\gamma_j]_3) &= \tilde{\delta}(a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3) \\ &= a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 + a_{p+1,j}[\gamma_{p+1}]_3 + \cdots + a_{p+q,j}[\gamma_{p+q}]_3 + \delta(v_j) \\ &= a_{1j}[\gamma_1]_3 + \cdots + a_{pj}[\gamma_p]_3 \end{aligned}$$

for any $1 \leq j \leq p$. Consider the map $s : \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^2) \rightarrow \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3)$ define by $\beta \mapsto \tilde{\delta} \circ \tilde{\beta}$. We can see that s is a homomorphism and is the required section. Hence the exact sequence (7) splits. \square

Observe the following lemma.

Lemma 5.5. *Let*

$$0 \rightarrow K \rightarrow G \rightarrow N \rightarrow 1$$

be a split extension of groups over N with K additive abelian group. For any g , there exist unique elements $k_g \in K$ and $n_g \in N$ such that $g = k_g n_g$. Then the map $k : G \rightarrow K$ defined by $g \mapsto k_g$ is a crossed homomorphism.

Since the proof is an easy exercise, we leave it to the leader. By applying this lemma to the split extension (7), we obtain the crossed homomorphism

$$(9) \quad \theta : \overline{\text{Aut}}(\mathcal{J}/\mathcal{J}^3) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathcal{J}), \text{gr}^2(\mathcal{J})).$$

5.2. Extensions of $\tilde{\eta}_1$.

In this subsection, we show that η_1 and η'_1 extend to $\text{Aut } G$ as crossed homomorphisms. According to the usual convention in homological algebra, for any group G and G -module M , we consider that G acts on M from the left if otherwise noted. Hence, the right actions mentioned above are read as the left one in the natural way. For example, for any $\sigma \in \text{Aut } G$ and $x \in G$, the left action of σ on $s_{ij}(x)$ is given by

$$\sigma \cdot s_{ij}(x) = s_{ij}(x^{\sigma^{-1}}).$$

Consider the associative algebra $\mathfrak{R}_{\mathbf{Q}}^m(G)$ and the filtration $\mathfrak{J}_G \supset \mathfrak{J}_G^2 \supset \mathfrak{J}_G^3 \supset \cdots$. By (9), we obtain the crossed homomorphism

$$\tilde{\theta}_{\mathcal{D}} : \overline{\text{Aut}}(\mathfrak{J}_G/\mathfrak{J}_G^3) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^2(\mathfrak{J}_G)).$$

For any $k \geq 2$, let $\rho_k : \text{Aut } G \rightarrow \overline{\text{Aut}}(\mathfrak{J}_G/\mathfrak{J}_G^k)$ be the natural homomorphism induced from the action of $\text{Aut } G$ on $\mathfrak{J}_G/\mathfrak{J}_G^k$. By composing $\tilde{\theta}_{\mathcal{D}}$ and ρ_2 , we obtain the crossed homomorphism

$$\theta_{\mathcal{D}} : \text{Aut } G \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^2(\mathfrak{J}_G)).$$

By observing (8), we can see that $\theta_{\mathcal{D}}$ is an extension of $\tilde{\eta}_1 : \mathcal{D}_G^m(1) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_G), \text{gr}^2(\mathfrak{J}_G))$.

6. ON THE $\text{SL}(m, \mathbf{C})$ -REPRESENTATION ALGEBRA OF F_n

In this section, we study the $\text{SL}(m, \mathbf{C})$ -representation algebra for free groups. Let F_n be the free group generated by x_1, x_2, \dots, x_n .

6.1. The graded quotients $\text{gr}^k(\mathfrak{J}_{F_n})$.

First, we give a basis of $\text{gr}^k(\mathfrak{J}_{F_n})$ as a \mathbf{Q} -vector space. For any $k \geq 1$, recall that (the coset classes of) the elements in

$$T_k := \left\{ \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \mid e_{ij, l} \geq 0, \sum_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \sum_{l=1}^n e_{ij, l} = k \right\} \subset \mathfrak{J}_{F_n}^k$$

generate $\text{gr}^k(\mathfrak{J}_{F_n})$ as a \mathbf{Q} -vector space. Then we have

Proposition 6.1. *For each $k \geq 1$, the set $T_k \pmod{\mathfrak{J}_{F_n}^{k+1}}$ forms a basis of $\text{gr}^k(\mathfrak{J}_{F_n})$ as a \mathbf{Q} -vector space.*

Proof. In order to show the linearly independentness of $T_k \pmod{\mathfrak{J}_{F_n}^{k+1}}$, assume

$$\sum' a(e_{ij, l}) \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \equiv 0 \pmod{\mathfrak{J}_{F_n}^{k+1}}$$

where the above sum \sum' runs over all tuples $(e_{ij, l})$ for $1 \leq i, j \leq m$, $(i, j) \neq (m, m)$ and $1 \leq l \leq n$ such that the sum of $e_{ij, l}$ s is equal to k . Denote by f the left hand side of the above equation, and assume $f \in \mathfrak{J}_{F_n}^{k+1}$. By observing the image of some $\text{SL}(m, \mathbf{C})$ -representations of F_n by f , we obtain all of $e_{ij, l}$ s are equal to zero as follows.

For any $n \geq 1$, consider

$$D_m := \{(z_{ij})_{1 \leq i, j \leq m} \in \mathbf{C}^{m^2} \mid \det((z_{ij})_{1 \leq i, j \leq m}) \neq 0\} \subset \mathbf{C}^{m^2}.$$

The set D_m is an open subset in \mathbf{C}^{m^2} . For any $1 \leq l \leq n$, take any $(z_{ij, l})_{1 \leq i, j \leq m-1} \in D_{m-1}$, and any $z_{im, l}, z_{mj, l} \in \mathbf{C}$ for $1 \leq i, j \leq m-1$. Define the representation $\rho : F_n \rightarrow \text{SL}(m, \mathbf{C})$ by

$$\rho(x_l) := (z_{ij, l})_{1 \leq i, j \leq m}$$

where for any $1 \leq l \leq n$, $z_{mm, l}$ is defined by

$$z_{mm, l} := Z_{mm, l}^{-1}(1 - (z_{1m, l}Z_{1m, l} + \dots + z_{m-1, m, l}Z_{m-1, m, l})),$$

and $Z_{ij, l}$ is the (i, j) -cofactor of the matrix $(z_{ij, l})_{1 \leq i, j \leq m}$. Note that

$$Z_{mm, l} = \det((z_{ij, l})_{1 \leq i, j \leq m-1}) \neq 0.$$

Then we have

$$f(\rho) = \sum' a(e_{ij,l}) \prod_{\substack{1 \leq i,j \leq m \\ (i,j) \neq (m,m)}} \prod_{l=1}^n (\bar{z}_{ij,l})^{e_{ij,l}}$$

where

$$\bar{z}_{ij,l} = \begin{cases} z_{ij,l}, & \text{if } i \neq j, \\ z_{ii,l} - 1, & \text{if } i = j. \end{cases}$$

On the other hand, from $f \in \mathfrak{J}_{F_n}^{k+1}$, we see that $f(\rho)$ can be written as a polynomial among $z_{ij,l}$ s with degree greater than k . Since we can take $(z_{ij,l})$ from the domain $D_{m-1} \times \mathbf{C}^{2m-2}$ arbitrary, we obtain $a(e_{ij,l}) = 0$ for any tuple $(e_{ij,l})$. This shows the linearly independentness of T_k . \square

As a corollary, we see

$$\dim_{\mathbf{Q}} \text{gr}^k(\mathfrak{J}_{F_n}) = \binom{(m^2 - 1)n + k - 1}{k}.$$

By exchanging $s_{11,l}$ for $s_{mm,l}$, from the same argument as Proposition 6.1, we see the following.

Corollary 6.2. *For each $k \geq 1$, the set*

$$T'_k := \left\{ \prod_{\substack{1 \leq i,j \leq m \\ (i,j) \neq (m,m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij,l}} \mid e_{ij,l} \geq 0, \sum_{\substack{1 \leq i,j \leq m \\ (i,j) \neq (1,1)}} \sum_{l=1}^n e_{ij,l} = k \right\} \subset \mathfrak{J}_{F_n}^k$$

forms a basis of $\text{gr}^k(\mathfrak{J}_{F_n})$ as a \mathbf{Q} -vector space.

Here we give a few problems.

Problem 6.3. (1) *Is $\bigcap_{k \geq 1} \mathfrak{J}_{F_n}^k$ is equal to 0?*

(2) *Is the algebra $\mathfrak{R}_{\mathbf{Q}}(F_n)$ isomorphic to SL_m -universal representation algebra of F_n over \mathbf{Q} ?*

In [31], we gave affirmative answers to the above questions for the SL_2 -representation case. It seems, however, quite difficult to apply the same method to the SL_m -representation case for $m \geq 3$ due to the combinatorial complexity.

Now, we review $\text{Aut } F_n$ and the IA-automorphism group $\text{IA}(F_n) = \mathcal{A}_{F_n}(1)$ of F_n . Let P, Q, S and U be automorphisms of F_n given by specifying its images of the basis x_1, \dots, x_n as follows:

	x_1	x_2	x_3	\cdots	x_{n-1}	x_n
P	x_2	x_1	x_3	\cdots	x_{n-1}	x_n
Q	x_2	x_3	x_4	\cdots	x_n	x_1
S	x_1^{-1}	x_2	x_3	\cdots	x_{n-1}	x_n
U	$x_1 x_2$	x_2	x_3	\cdots	x_{n-1}	x_n

In 1924, Nielsen [23] showed that $\text{Aut } F_n$ is generated by P, Q, S and U , and gave finitely many relators among them. On the other hand, Magnus [18] showed that for

any $n \geq 3$, $\text{IA}(F_n)$ is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $1 \leq i, j \leq n$, and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i[x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $1 \leq i, j, l \leq n$ and $j < l$. Let H be the abelianization $F_n/\Gamma_{F_n}(2)$ of F_n . By observing the images of Nielsen's generators, we can see that the natural homomorphism $\text{Aut } F_n \rightarrow \text{Aut } H$ is surjective. By fixing the basis of the free abelian group H induced from that x_1, \dots, x_n of F_n , we can identify $\text{Aut}(H)$ with the general linear group $\text{GL}(n, \mathbf{Z})$. Hence we have $\text{Aut } F_n/\text{IA}(F_n) \cong \text{GL}(n, \mathbf{Z})$.

In [31], we showed that $\mathcal{D}_{F_n}^2(k) = \mathcal{A}_{F_n}(k)$ for $1 \leq k \leq 4$. Hence, from Theorem 4.10, we have the following.

Theorem 6.4. *For any $m \geq 2$, we have $\mathcal{D}_{F_n}^m(k) = \mathcal{A}_{F_n}(k)$ for any $1 \leq k \leq 4$.*

In particular, we see that $\text{GL}(n, \mathbf{Z}) \cong \text{Aut } F_n/\mathcal{D}_{F_n}^m(1)$ naturally acts on $\text{gr}^k(\mathfrak{J}_{F_n})$ for any $k \geq 1$.

Proposition 6.5. *For any $n \geq 2$ and $k \geq 1$, we have*

$$\text{gr}^k(\mathfrak{J}_{F_n}) \cong \bigoplus'_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \bigotimes S^{e_{ij}} H_{\mathbf{Q}}$$

as a $\text{GL}(n, \mathbf{Z})$ -module. Here the sum runs over all tuples (e_{ij}) for $1 \leq i, j \leq m$ and $(i, j) \neq (m, m)$ such that the sum of the e_{ij} is equal to k .

Proof. Let \mathfrak{M} be the right hand side of the above equation. First, for any $1 \leq i, j \leq m$ and $e \geq 1$, the homomorphism $f_{ij}^e : S^e H_{\mathbf{Q}} \rightarrow \text{gr}^e(\mathfrak{J}_{F_n})$ defined by

$$x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} \mapsto s_{ij}(x_1)^{l_1} s_{ij}(x_2)^{l_2} \cdots s_{ij}(x_n)^{l_n} \pmod{\mathfrak{J}_{F_n}^{e+1}}$$

for $l_1 + l_2 + \cdots + l_n = e$ is $\text{Aut } F_n$ -equivariant. In fact, for any Nielsen generators $\sigma = P, Q, S$ and U of $\text{Aut } F_n$, we can check $f_{ij}^e(x^\sigma) = (f_{ij}^e(x))^\sigma$ for any $x = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$. For example, we see

$$\begin{aligned} f_{ij}^e((x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n})^U) &= f_{ij}^e((x_1 + x_2)^{l_1} x_2^{l_2} \cdots x_n^{l_n}) \\ &= f_{ij}^e\left(\sum_{t=0}^{l_1} \binom{l_1}{t} x_1^t x_2^{l_1-t} x_2^{l_2} \cdots x_n^{l_n}\right) \\ &\equiv \sum_{t=0}^{l_1} \binom{l_1}{t} s_{ij}(x_1)^t s_{ij}(x_2)^{l_1-t+l_2} \cdots s_{ij}(x_n)^{l_n} \pmod{\mathfrak{J}_{F_n}^{e+1}} \\ &\equiv s_{ij}(x_1 x_2)^{l_1} s_{ij}(x_2)^{l_2} \cdots s_{ij}(x_n)^{l_n} \pmod{\mathfrak{J}_{F_n}^{e+1}} \\ &= (f_{ij}^e(x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}))^U. \end{aligned}$$

Since $\text{IA}(F_n)$ trivially acts on both of $S^e H_{\mathbf{Q}}$ and $\text{gr}^e(\mathfrak{J}_{F_n})$, we obtain the surjective $\text{GL}(n, \mathbf{Z})$ -equivariant homomorphism $F : \mathfrak{M} \rightarrow \text{gr}^k(\mathfrak{J}_{F_n})$ defined by

$$\begin{aligned} & \sum' a_{e_{11}, \dots, e_{mm-1}} \bigotimes_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} X_{e_{ij}} \\ & \mapsto \sum' a_{e_{11}, \dots, e_{mm-1}} \bigotimes_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} f_{ij}^{e_{ij}}(X_{e_{ij}}) \end{aligned}$$

for any $X_{e_{ij}} \in S^{e_{ij}} H_{\mathbf{Q}}$ and $a_{e_{11}, \dots, e_{mm-1}} \in \mathbf{Q}$. Here the sum runs over all tuples (e_{ij}) for $1 \leq i, j \leq m$ and $(i, j) \neq (m, m)$ such that the sum of e_{ij} s is equal to k . The surjectivity of F follows from Proposition 6.1. In fact, for any element

$$Y := \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}}$$

in the basis T_k of $\text{gr}^k(\mathfrak{J}_{F_n})$, set

$$X := \sum'' \bigotimes_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} x_1^{e_{ij, 1}} \cdots x_n^{e_{ij, n}} \in \mathfrak{M}$$

where the sum runs over all tuples $(e_{ij, l})$ for $1 \leq i, j \leq m$, $(i, j) \neq (m, m)$ and $1 \leq l \leq n$ such that the sum of $e_{ij, l}$ s is equal to k . Then we have $Y = F(X)$.

Next, we prove that F is an isomorphism by showing that the dimensions of \mathfrak{M} and $\text{gr}^k(\mathfrak{J}_{F_n})$ as \mathbf{Q} -vector spaces are equal. The basis T_k can be rewritten as

$$\begin{aligned} T_k &= \left\{ \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \prod_{l=1}^n s_{ij}(x_l)^{e_{ij, l}} \mid e_{ij, l} \geq 0, \sum_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} \sum_{l=1}^n e_{ij, l} = k \right\} \\ &= \left\{ \prod_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} s_{ij}(x_1)^{e_{ij, 1}} \cdots s_{ij}(x_n)^{e_{ij, n}} \mid \sum_{l=1}^n e_{ij, l} = e_{ij}, \sum_{\substack{1 \leq i, j \leq m \\ (i, j) \neq (m, m)}} e_{ij} = k \right\} \end{aligned}$$

From the last term of the above equation, we see that the number of elements in T_k is equal to $\dim_{\mathbf{Q}} \mathfrak{M}$. This completes the proof of Proposition 6.5. \square

6.2. On the extension of $\tilde{\eta}_1$.

Let us consider the homomorphism η_1 and its extension to $\text{Aut } F_n$ as a crossed homomorphism. Observe

$$\begin{aligned} s_{ij}(x[y, z]) - s_{ij}(x) &= s_{ij}(x) + s_{ij}([y, z]) + \sum_{k=1}^m s_{ik}(x) s_{kj}([y, z]) - s_{ij}(x) \\ &\equiv s_{ij}([y, z]) \pmod{\mathfrak{J}_{F_n}^3} \\ &\equiv \sum_{k=1}^m (s_{ik}(y) s_{kj}(z) - s_{ik}(z) s_{kj}(y)) \pmod{\mathfrak{J}_{F_n}^3} \end{aligned}$$

for any $x, y, z \in F_n$. The last equality is obtained from the straightforward calculation with (3). By using this, we can easily calculate the images of Magnus generators K_{ij} and K_{ijl} of $\text{IA}(F_n)$ by η_1 as follows. From Proposition 6.1, we see that elements $s_{ij}(x_l)$ for $(i, j) \neq (m, m)$ and $1 \leq l \leq n$ form a basis of $\text{gr}^1(\mathfrak{J}_{F_n})$. Let $s_{ij}(x_l)^*$ be its dual basis of $\text{gr}^1(\mathfrak{J}_{F_n})^* := \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \mathbf{Q})$. Then we have

$$(10) \quad \begin{aligned} \tilde{\eta}_1(K_{ij}) &= \sum_{\substack{1 \leq p, q \leq m \\ (p, q) \neq (m, m)}} \sum_{k=1}^m s_{pq}(x_i)^* \otimes (s_{pk}(x_i)s_{kq}(x_j) - s_{pk}(x_j)s_{kq}(x_i)), \\ \tilde{\eta}_1(K_{ijl}) &= \sum_{\substack{1 \leq p, q \leq m \\ (p, q) \neq (m, m)}} \sum_{k=1}^m s_{pq}(x_i)^* \otimes (s_{pk}(x_j)s_{kq}(x_l) - s_{pk}(x_l)s_{kq}(x_j)). \end{aligned}$$

From the argument in Subsection 5.2, we have the crossed homomorphism

$$\tilde{\theta}_{\mathcal{D}} : \overline{\text{Aut}}(\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^3) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n})).$$

On the other hand, the natural action of $\text{Aut } F_n$ on $\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^3$ induces the homomorphism $\text{Aut } F_n \rightarrow \overline{\text{Aut}}(\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^3)$. By composing this map and $\tilde{\theta}_{\mathcal{D}}$, we obtain the crossed homomorphism

$$\theta_{F_n} : \text{Aut } F_n \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n})),$$

which is an extension of $\tilde{\eta}_1$ to $\text{Aut } F_n$. In this subsection, we study this crossed homomorphism and relations to f_K and f_M .

To begin with, we determine the images of Nielsen's generators by θ_{F_n} . Let $\rho_k : \text{Aut } F_n \rightarrow \overline{\text{Aut}}(\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^k)$ be the homomorphism induced from the action of $\text{Aut } F_n$ on $\mathfrak{J}_{F_n}/\mathfrak{J}_{F_n}^k$. By direct computation, for $\sigma = P, Q$, we can see that $\theta_{F_n}(\sigma) = 0$ since $\rho_3(\sigma)s(\rho_2(\sigma^{-1}))$ satisfies

$$[s_{ij}(x_l)]_3 \mapsto [s_{ij}(x_l)]_3$$

for any $1 \leq i, j \leq m$ and $1 \leq l \leq n$. Consider $\rho_3(S)s(\rho_2(S^{-1}))$. By using (3), for any $1 \leq i, j \leq m$, we see

$$\begin{aligned} s(\rho_2(S^{-1}))([s_{ij}(x_1)]_3) &= -[s_{ij}(x_1)]_3, \\ \rho_3(S)(-[s_{ij}(x_1)]_3) &= -[s_{ij}(x_1^{-1})]_3 = [s_{ij}(x_1)]_3 - \sum_{k=1}^m [s_{ik}(x_1)s_{kj}(x_1)]_3. \end{aligned}$$

Hence we obtain

$$\theta_{F_n}(S) = - \sum_{(i, j) \neq (m, m)} \sum_{k=1}^m s_{ij}(x_1)^* \otimes [s_{ik}(x_1)s_{kj}(x_1)]_3.$$

where $s_{ij}(x_l)^*$ means the dual basis in $\text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \mathbf{Q})$ of $s_{ij}(x_l)$. Similarly, we can obtain

$$\theta_{F_n}(U) = - \sum_{(i, j) \neq (m, m)} \sum_{k=1}^m s_{ij}(x_1)^* \otimes ([s_{ik}(x_2)s_{kj}(x_2) + [s_{ik}(x_1)s_{kj}(x_2)]_3).$$

In the following, we introduce the crossed homomorphism $f_1 : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}$ and $f_2 : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}$ from θ_{F_n} , and study relations between them and f_K and f_M .

First, we review Kawazumi's cocycle $f_K : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}$ and Morita's cocycle $f_M : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}$. The images of Nielsen's generators of $\text{Aut } F_n$ by f_K and f_M are given by

$$f_K(\sigma) = \begin{cases} -x_1^* \otimes x_1 \wedge x_2, & \sigma = U, \\ 0, & \sigma = P, Q, S, \end{cases} \quad f_M(\sigma) = \begin{cases} -x_1, & \sigma = S, \\ 0, & \sigma = P, Q, U. \end{cases}$$

(For details, see [24] and [32].)

Now, recall that

$$\text{gr}^1(\mathfrak{J}_{F_n}) \cong H_{\mathbf{Q}}^{\oplus(m^2-1)}, \quad \text{gr}^2(\mathfrak{J}_{F_n}) \cong (S^2 H_{\mathbf{Q}})^{\oplus(m^2-1)} \oplus (H_{\mathbf{Q}}^{\otimes 2})^{\oplus \binom{m^2-1}{2}},$$

and that $\{s_{ij}(x_l) \mid (i, j) \neq (m, m), \ 1 \leq l \leq n\}$ and T_2 are basis of them respectively. Let $p_1 : \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n})) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \text{gr}^2(\mathfrak{J}_{F_n}))$ be the homomorphism induced from the inclusion map $H_{\mathbf{Q}} \rightarrow \text{gr}^1(\mathfrak{J}_{F_n})$ defined by

$$\sum_{l=1}^n c_l x_l \mapsto \sum_{l=1}^n c_l s_{11}(x_l).$$

Let $p_2 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \text{gr}^2(\mathfrak{J}_{F_n})) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2})$ be the homomorphism induced from the projection $\text{gr}^2(\mathfrak{J}_{F_n}) \rightarrow H_{\mathbf{Q}}^{\otimes 2}$ defined by

$$\sum_{t \in T_2} c_t t \mapsto \sum_{1 \leq i < j \leq n} c_{s_{12}(x_i) s_{21}(x_j)} x_i \otimes x_j.$$

Let $p_3 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, S^2 H_{\mathbf{Q}}) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2})$ be the homomorphism induced from the homomorphism $H_{\mathbf{Q}}^{\otimes 2} \rightarrow S^2 H_{\mathbf{Q}}$ defined by

$$x_i \otimes x_j \mapsto x_i \wedge x_j$$

for any $1 \leq i < j \leq n$. Then the composition map $p_3 \circ p_2 \circ p_1$, denoted by p , is an $\text{Aut } F_n$ -equivariant homomorphism. Set

$$f_1 := p \circ \theta_{F_n} : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}.$$

On the other hand, Let $q_2 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \text{gr}^2(\mathfrak{J}_{F_n})) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, S^2 H_{\mathbf{Q}})$ be the homomorphism induced from the projection $\text{gr}^2(\mathfrak{J}_{F_n}) \rightarrow S^2 H_{\mathbf{Q}}$ defined by

$$\sum_{t \in T_2} c_t t \mapsto \sum_{1 \leq i < j \leq n} c_{s_{11}(x_i) s_{11}(x_j)} x_i x_j.$$

Let $q_3 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, S^2 H_{\mathbf{Q}}) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2})$ be the homomorphism induced from the homomorphism $S^2 H_{\mathbf{Q}} \rightarrow H_{\mathbf{Q}}^{\otimes 2}$ defined by

$$x_i x_j \mapsto x_i \otimes x_j + x_j \otimes x_i$$

for any $1 \leq i < j \leq n$. Let $q_4 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2}) \cong H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} H_{\mathbf{Q}}^{\otimes 2} \rightarrow H_{\mathbf{Q}}$ be the contraction map with respect to the first and the second component. Then the composition map $q_4 \circ q_3 \circ q_2 \circ p_1$, denoted by q , is an $\text{Aut } F_n$ -equivariant homomorphism. Set

$$f_2 := q \circ \theta_{F_n} : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}.$$

Finally, set

$$x := x_1 + x_2 + \cdots + x_n \in H_{\mathbf{Q}},$$

and let δ_x be the principal crossed homomorphism associated to x . Then we obtain the following.

Theorem 6.6. *For any $n \geq 2$,*

$$\begin{aligned} f_K &= f_1, \\ f_M &= -f_2 + \delta_x \end{aligned}$$

as crossed homomorphisms.

Proof. Since

$$f_1(\sigma) = \begin{cases} -x_1^* \otimes x_1 \wedge x_2, & \sigma = U, \\ 0, & \sigma = P, Q, S, \end{cases} \quad f_2(\sigma) = \begin{cases} 0, & \sigma = P, Q, \\ -x_1, & \sigma = S, \\ -x_2, & \sigma = U, \end{cases}$$

we can obtain the required results by direct computation. \square

Since f_K and f_M define non-trivial cohomology classes in $H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda H_{\mathbf{Q}}) = \mathbf{Q}^{\otimes 2}$ and $H^1(\text{Aut } F_n, H_{\mathbf{Q}}) = \mathbf{Q}$ by [24] and [27] respectively, we see that θ_{F_n} defines non-zero cohomology class in $H^1(\text{Aut } F_n, \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n})))$. We remark that in our forthcoming paper [33], we computed

$$H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} S^2 H_{\mathbf{Q}}) \cong \mathbf{Q}.$$

Thus we have

$$\begin{aligned} & H^1(\text{Aut } F_n, \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_{F_n}), \text{gr}^2(\mathfrak{J}_{F_n}))) \\ & \cong H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} S^2 H_{\mathbf{Q}})^{\oplus (m^2-1)^2} \\ & \quad \bigoplus H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} H_{\mathbf{Q}}^{\otimes 2})^{\oplus (m^2-1) \binom{m^2-1}{2}} \\ & \cong \mathbf{Q}^{\oplus \frac{1}{2}(m^2-1)^2(3m^2-4)}. \end{aligned}$$

7. ON THE $\text{SL}(m, \mathbf{C})$ -REPRESENTATION ALGEBRA OF H

In this section, we study the $\text{SL}(m, \mathbf{C})$ -representation algebra for free abelian groups. Recall that H is the free abelian group of rank n with basis x_1, \dots, x_n . It seems to be quite difficult to obtain a basis of $\text{gr}^k(\mathfrak{J}_H)$ for a general $k \geq 1$. In this section, we give basis of $\text{gr}^k(\mathfrak{J}_H)$ for $1 \leq k \leq 2$. Then, by using it, we study the extension of η_1 .

By Lemma 3.4, we have the surjective homomorphism

$$\bar{\mathbf{a}} : \mathfrak{R}_{\mathbf{Q}}^m(F_n) \rightarrow \mathfrak{R}_{\mathbf{Q}}^m(H)$$

induced from the abelianization $\mathbf{a} : F_n \rightarrow H$. The above map $\bar{\mathbf{a}}$ also induces surjective homomorphisms

$$\bar{\mathbf{a}}^k : \text{gr}^k(\mathfrak{J}_{F_n}) \rightarrow \text{gr}^k(\mathfrak{J}_H)$$

for any $k \geq 1$. We study the kernel of $\bar{\mathbf{a}}^k$ for $1 \leq k \leq 2$.

In this section, we denote by $F_{ij}(z)$ the $m \times m$ -matrix whose (i, j) -entry is $z \in \mathbf{C}$ and the other entries zero. Furthermore, we denote by $E_i(z) \in \text{SL}(m, \mathbf{C})$ the matrix whose (i, i) -entry is $z \in \mathbf{C} \setminus \{0\}$, the other diagonal entries one, and the other entries zero.

7.1. The graded quotients $\text{gr}^1(\mathfrak{J}_H)$.

First, we show that $\bar{\mathfrak{a}}^1$ is isomorphism. From Corollary 6.2 and Proposition 6.5, we have $\text{gr}^1(\mathfrak{J}_{F_n}) \cong H_{\mathbf{Q}}^{\oplus m^2-1}$ and

$$T_1 = \{s_{ij}(x_l) \mid 1 \leq i, j \leq m, (i, j) \neq (m, m), 1 \leq l \leq n\} \pmod{\mathfrak{J}_{F_n}^2}$$

forms a basis of $\text{gr}^1(\mathfrak{J}_{F_n})$. For each $1 \leq i, j \leq m$ and $1 \leq l \leq n$, set $s_{ij}(\bar{x}_l) := \bar{\mathfrak{a}}(s_{ij}(x_l)) = s_{ij}(\mathfrak{a}(x_l))$, and

$$\bar{T}_1 := \{s_{ij}(\bar{x}_l) \mid 1 \leq i, j \leq m, (i, j) \neq (m, m), 1 \leq l \leq n\} \subset \mathfrak{R}_{\mathbf{Q}}^m(H).$$

Proposition 7.1. *The set $\bar{T}_1 \pmod{\mathfrak{J}_H^2}$ forms a basis of $\text{gr}^1(\mathfrak{J}_H)$ as a \mathbf{Q} -vector space.*

Proof. It suffices to show that the linearly independentness of elements in $\bar{T}_1 \pmod{\mathfrak{J}_H^2}$. Set

$$f := \sum_{(i,j) \neq (m,m)} \sum_{1 \leq l \leq n} a_{ij,l} s_{ij}(\bar{x}_l) \in \mathfrak{R}_{\mathbf{Q}}^m(H)$$

for $a_{ij,l} \in \mathbf{Q}$, and assume that $f \equiv 0 \pmod{\mathfrak{J}_H^2}$. Take any $(i, j) \neq (m, m)$, and fix it.

If $i \neq j$, take any $z_l \in \mathbf{C}$ for $1 \leq l \leq n$, and consider the representation $\rho_1 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined by

$$x_l \mapsto E_n + F_{ij}(z_l).$$

Then we see that

$$f(\rho) = \sum_{1 \leq l \leq n} a_{ij,l} z_l + (\text{terms of degree} \geq 2).$$

The assumption $f \equiv 0 \pmod{\mathfrak{J}_H^2}$ means

$$\sum_{1 \leq l \leq n} a_{ij,l} z_l = \text{terms of degree} \geq 2.$$

Since we can take $z_l \in \mathbf{C}$ arbitrary, we see that $a_{ij,l} = 0$ for any $1 \leq l \leq n$.

If $i = j$, take any $z_l \in D := \{z \in \mathbf{C} \mid |z| < 1\}$ for $1 \leq l \leq n$, and consider the representation $\rho_2 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined by

$$x_l \mapsto E_i(1 + z_l)E_m((1 + z_l)^{-1}).$$

Then by the same argument as above, we see $a_{ii,l} = 0$ for any $1 \leq l \leq n$. Thus we obtain the required result. \square

7.2. The graded quotients $\text{gr}^2(\mathfrak{J}_H)$.

Next we consider the case of $\text{gr}^2(\mathfrak{J}_H)$. The basic idea of the strategy to find a basis of $\text{gr}^2(\mathfrak{J}_H)$ is the same as that of $\text{gr}^1(\mathfrak{J}_H)$. From Corollary 6.2 and Proposition 6.5, we have

$$\text{gr}^2(\mathfrak{J}_{F_n}) \cong (S^2 H_{\mathbf{Q}})^{\oplus m^2-1} \oplus (H_{\mathbf{Q}}^{\otimes 2})^{\oplus \binom{m^2-1}{2}}$$

and

$$\begin{aligned} T_2 = & \{s_{ij}(x_p)s_{ij}(x_q) \mid 1 \leq i, j \leq m, (i, j) \neq (m, m), 1 \leq p \leq q \leq n\} \\ & \cup \{s_{ij}(x_p)s_{hk}(x_q) \mid (i, j) <_{\text{lex}} (h, k), (i, j), (h, k) \neq (m, m), 1 \leq p, q \leq n\} \end{aligned}$$

$(\text{mod } \mathfrak{J}_{F_n}^3)$ forms a basis of $\text{gr}^2(\mathfrak{J}_{F_n})$. We rewrite this basis. Set

$$\begin{aligned} t_{ij,hk}(p, q) &:= s_{ij}(x_p)s_{hk}(x_q) - s_{ij}(x_q)s_{hk}(x_p), \\ u_{ij,hk}(p, q) &:= s_{ij}(x_p)s_{hk}(x_q) + s_{ij}(x_q)s_{hk}(x_p), \\ v_{ij}(p, q) &:= s_{ij}(x_p)s_{ij}(x_q). \end{aligned}$$

Then

$$\begin{aligned} T'_2 &:= \{t_{ij,hk}(p, q) \mid (i, j) <_{\text{lex}} (h, k), (i, j), (h, k) \neq (m, m), p < q\} \\ &\cup \{u_{ij,hk}(p, q) \mid (i, j) \leq_{\text{lex}} (h, k), (i, j), (h, k) \neq (m, m), p \leq q\} \\ &\cup \{v_{ij}(p, q) \mid (i, j) \neq (m, m)\} \pmod{\mathfrak{J}_{F_n}^3} \end{aligned}$$

also forms a basis of $\text{gr}^2(\mathfrak{J}_{F_n})$.

Now, we observe some elements in the kernel of $\bar{\alpha}^2$. For any $1 \leq i, j \leq m$, and any $1 \leq p, q \leq n$, the equation $\bar{\alpha}(s_{ij}(x_p x_q)) = \bar{\alpha}(s_{ij}(x_q x_p))$ induces the fact that

$$R_{ij}(p, q) := \sum_{k=1}^m t_{ik,kj}(p, q) \pmod{\mathfrak{J}_{F_n}^3} \in \text{Ker}(\bar{\alpha}^2).$$

More precisely, we have

$$\begin{aligned} R_{ij}(p, q) &= -t_{1j,i1}(p, q) - \cdots - t_{i-1j,ii-1}(p, q) + t_{ii,ij}(p, q) + \cdots + t_{im,mj}(p, q), \quad (i \neq 1), \\ R_{1j}(p, q) &= t_{11,1j}(p, q) + \cdots + t_{1m,mj}(p, q), \quad (j \neq 1). \end{aligned}$$

Set

$$\begin{aligned} \bar{t}_{ij,hk}(p, q) &:= \bar{\alpha}(t_{ij,hk}(p, q)), \quad \bar{u}_{ij,hk}(p, q) := \bar{\alpha}(u_{ij,hk}(p, q)), \\ \bar{v}_{ij}(p, q) &:= \bar{\alpha}(v_{ij}(p, q)). \end{aligned}$$

From the above observation, we see that we can remove $\bar{t}_{1j,i1}(p, q)$, $\bar{t}_{11,i1}(p, q)$ and $\bar{t}_{11,1j}(p, q)$ for $i, j \neq 1$ from the generating set $\alpha^2(T'_2)$ of $\text{gr}^2(\mathfrak{J}_H)$ by using $R_{ij}(p, q)$ and $R_{1j}(p, q)$ respectively. Define the index sets I, J by

$$\begin{aligned} I &:= \{(i, j, h, k) \mid (i, j) < (h, k), (i, j), (h, k) \neq (m, m)\}, \\ J &:= I \setminus \{(1, j, i, 1), (1, 1, i, 1), (1, 1, 1, j) \mid i, j \neq 1\}. \end{aligned}$$

Set

$$\begin{aligned} Y &:= \{\bar{t}_{ij,hk}(p, q) \mid (i, j, h, k) \in J, p < q\} \cup \{\bar{u}_{ij,hk}(p, q) \mid (i, j, h, k) \in I, p \leq q\} \\ &\cup \{\bar{v}_{ij}(p, q) \mid (i, j) \neq (m, m), p \leq q\}. \end{aligned}$$

Proposition 7.2. *The set $Y \pmod{\mathfrak{J}_H^3}$ forms a basis of $\text{gr}^2(\mathfrak{J}_H)$ as a \mathbf{Q} -vector space.*

Proof. It suffices to show the linearly independentness of Y . Set

$$\begin{aligned} f &:= \sum_{(i,j,h,k) \in J} \sum_{p < q} \alpha_{ij,hk}(p, q) \bar{t}_{ij,hk}(p, q) + \sum_{(i,j,h,k) \in I} \sum_{p \leq q} \beta_{ij,hk}(p, q) \bar{u}_{ij,hk}(p, q) \\ &\quad + \sum_{(i,j) \neq (m,m)} \sum_{p \leq q} \gamma_{ij}(p, q) \bar{v}_{ij}(p, q) \in \mathfrak{R}_{\mathbf{Q}}^m(H) \end{aligned}$$

for $\alpha_{ij,hk}(p, q), \beta_{ij,hk}(p, q), \gamma_{ij}(p, q) \in \mathbf{Q}$, and assume $f \equiv 0$.

Step 1. The proof of $\gamma_{ij}(p, q) = 0$.

Take any $(i, j) \neq (m, m)$, and fix it. If $i = j$, by using the representation $\rho_1 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined in the proof of Proposition 7.1, and by using $f \equiv 0 \pmod{(\text{mod } \mathfrak{J}_H^3)}$, we see that

$$\sum_{p \leq q} \gamma_{ij}(p, q) z_p z_q = \text{terms of degree } \geq 3.$$

Since we can take $z_l \in \mathbf{C}$ arbitrary, we see that $\gamma_{ij}(p, q) = 0$ for any $1 \leq p \leq q \leq n$. Similarly, if $i = j$, by considering the representation $\rho_2 : H \rightarrow \text{SL}(m, \mathbf{C})$, we can obtain $\gamma_{ii}(p, q) = 0$ for any $1 \leq p \leq q \leq n$.

Step 2. The proof of $\beta_{ij}(p, q) = 0$.

For any $1 \leq l \leq n$, take any $z_{l1}, \dots, z_{lm-1} \in \mathbf{C} \setminus \{0\}$, and set $z_{lm} := (z_{l1} \cdots z_{lm-1})^{-1}$. For any $B = (b_{ij}) \in \text{GL}(m, \mathbf{C})$, consider the representation $\rho_3 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined by

$$x_l \mapsto B \begin{pmatrix} z_{l1} & & \\ & \ddots & \\ & & z_{lm} \end{pmatrix} \tilde{B}$$

for any $1 \leq l \leq n$ where $\tilde{B} = (\tilde{b}_{ji})$ is the adjugate matrix of B . By considering the power series expansion among $z_{lr} - 1$, we can see that for any $1 \leq l \leq n$, the (i, j) -entry of $\rho_3(x_l) - E_m$ is given by

$$\frac{1}{|B|} \left(\sum_{r=1}^{m-1} (b_{ir} \tilde{b}_{jr} - b_{im} \tilde{b}_{jm})(z_{lr} - 1) + (\text{terms of degree } \geq 2) \right).$$

Hence we have

$$\begin{aligned} f(\rho_3) = \frac{1}{|B|^2} & \left\{ \sum_{(i,j,h,k) \in J} \sum_{p < q} \alpha_{ij,hk}(p, q) \sum_{r,s=1}^{m-1} \left((b_{ir} \tilde{b}_{jr} - b_{im} \tilde{b}_{jm})(b_{hs} \tilde{b}_{ks} - b_{hm} \tilde{b}_{km}) \right. \right. \\ & \quad \left. \left. - (b_{is} \tilde{b}_{js} - b_{im} \tilde{b}_{jm})(b_{hr} \tilde{b}_{kr} - b_{hm} \tilde{b}_{km}) \right) (z_{pr} - 1)(z_{qs} - 1) \right. \\ & + \sum_{(i,j,h,k) \in I} \sum_{p \leq q} \beta_{ij,hk}(p, q) \sum_{r,s=1}^{m-1} \left((b_{ir} \tilde{b}_{jr} - b_{im} \tilde{b}_{jm})(b_{hs} \tilde{b}_{ks} - b_{hm} \tilde{b}_{km}) \right. \\ & \quad \left. \left. + (b_{is} \tilde{b}_{js} - b_{im} \tilde{b}_{jm})(b_{hr} \tilde{b}_{kr} - b_{hm} \tilde{b}_{km}) \right) (z_{pr} - 1)(z_{qs} - 1) \right\} \\ & + (\text{terms of degree } \geq 3). \end{aligned}$$

Since $f \equiv 0 \pmod{(\text{mod } \mathfrak{J}_H^3)}$, we see that each coefficient of $(z_{pr} - 1)(z_{qs} - 1)$ is equal to zero. In particular, for any $p \leq q$, by observing the coefficient of $(z_{p1} - 1)(z_{q1} - 1)$, we have

$$\sum_{(i,j,h,k) \in I} 2\beta_{ij,hk}(p, q) (b_{i1} \tilde{b}_{j1} - b_{im} \tilde{b}_{jm})(b_{h1} \tilde{b}_{k1} - b_{hm} \tilde{b}_{km}) = 0.$$

Now we consider the linear ordering on the monomials among b_{ij} induced from the lexicographic ordering on $\{(i, j) \mid 1 \leq i, j \leq m\}$. Then the minimum element in $(b_{i1} \tilde{b}_{j1} - b_{im} \tilde{b}_{jm})$ is

$$b_{im} b_{11} \cdots b_{j-1j-1} b_{j+1j} \cdots b_{mm-1}$$

coming from $b_{im}\tilde{b}_{jm}$, and that in $(b_{h1}\tilde{b}_{k1} - b_{hm}\tilde{b}_{km})$ is

$$b_{hm}b_{11} \cdots b_{k-1k-1}b_{k+1k} \cdots b_{mm-1}$$

coming from $b_{hm}\tilde{b}_{km}$. Thus since we can take $B = (b_{ij}) \in \text{GL}(m, \mathbf{C})$ arbitrary, by observing the coefficient of the minimum element

$$b_{im}b_{11} \cdots b_{j-1j-1}b_{j+1j} \cdots b_{mm-1} \cdot b_{hm}b_{11} \cdots b_{k-1k-1}b_{k+1k} \cdots b_{mm-1}$$

we obtain $\beta_{ij,hk}(p, q) = 0$.

Step 3. The proof of $\alpha_{ij}(p, q) = 0$.

Take any $(i, j, h, k) \in J$, and set $N := \#\{i, j, h, k\}$.

(1) The case of $N = 4$.

For any $1 \leq l \leq n$, take any $z_l, w_l \in \mathbf{C}$, and consider the representation $\rho_4 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined by

$$x_l \mapsto E_m + F_{ij}(z_l) + F_{hk}(w_l).$$

Then by an argument similar to the above, we see $\alpha_{ij,hk}(p, q) = 0$ for any $p \leq q$.

(2) The case of $N \leq 3$.

We have to consider the following cases of indices:

(i)	(i, i, j, h)	(ii)	(i, j, i, h)	(iii)	(i, j, h, i)
(iv)	(i, j, j, h)	(v)	(i, j, h, j)	(vi)	(i, j, h, h)
(vii)	(i, i, i, j)	(viii)	(i, i, j, i)	(ix)	(i, j, i, i)
(x)	(i, j, j, j)	(xi)	(i, i, j, j)	(xii)	(i, j, j, i)

(ii) and (v). For any $1 \leq l \leq n$, take any $z_l, w_l \in \mathbf{C}$. In the case where $(i, j, i, h) \in J$, define the representation $\rho_5 : H \rightarrow \text{SL}(m, \mathbf{C})$ by

$$x_l \mapsto E_m + F_{ij}(z_l) + F_{ih}(w_l).$$

Then we can see $\alpha_{ij,ih}(p, q) = 0$ for any $p \leq q$. Similarly, in the case where $(i, j, h, j) \in J$, from the representation $\rho_6 : H \rightarrow \text{SL}(m, \mathbf{C})$ defined by

$$x_l \mapsto E_m + F_{ij}(z_l) + F_{hj}(w_l),$$

we can obtain $\alpha_{ij,hj}(p, q) = 0$ for any $p \leq q$.

Next, we consider the other cases. For any $1 \leq l \leq n$, take any $z_l, w_l \in D$. For any $b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32} \in \mathbf{C}$ such that $\Delta := b_{12}b_{23}b_{31} + b_{13}b_{32}b_{21} \neq 0$, define the 3×3 matrix C by

$$\begin{aligned} C_l &:= \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix} \begin{pmatrix} z_l & 0 & 0 \\ 0 & w_l & 0 \\ 0 & 0 & (z_l w_l)^{-1} \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}^{-1} \\ &= \frac{1}{\Delta} \begin{pmatrix} b_{12}b_{23}b_{31}w_l + b_{13}b_{32}b_{21}(z_l w_l)^{-1} & -b_{12}b_{13}b_{31}(w_l - (z_l w_l)^{-1}) & b_{12}b_{13}b_{21}(w_l - (z_l w_l)^{-1}) \\ b_{21}b_{23}b_{32}(-z_l + (z_l w_l)^{-1}) & b_{12}b_{23}b_{31}(z_l w_l)^{-1} + b_{13}b_{32}b_{21}z_l & b_{12}b_{21}b_{23}(z_l - (z_l w_l)^{-1}) \\ b_{23}b_{31}b_{32}(-z_l + w_l) & b_{13}b_{31}b_{32}(z_l - w_l) & b_{12}b_{23}b_{31}z_l + b_{13}b_{32}b_{21}w_l \end{pmatrix}. \end{aligned}$$

By observing the power series expansion of each entry of $C_l - E_3$ with respect to $z'_l := z_l - 1$ and $w'_l := w_l - 1$, and by writing down the degree one part of each entry,

we have

$$\frac{1}{\Delta} \begin{pmatrix} b_{12}b_{23}b_{31}w'_l - b_{13}b_{32}b_{21}(z'_l + w'_l) & -b_{12}b_{13}b_{31}(z'_l + 2w'_l) & b_{12}b_{13}b_{21}(z'_l + 2w'_l) \\ -b_{21}b_{23}b_{32}(w'_l + 2z'_l) & -b_{12}b_{23}b_{31}(z'_l + w'_l) + b_{13}b_{32}b_{21}z'_l & b_{12}b_{21}b_{23}(2z'_l + w'_l) \\ b_{23}b_{31}b_{32}(-z'_l + w'_l) & b_{13}b_{31}b_{32}(z'_l - w'_l) & b_{12}b_{23}b_{31}z'_l + b_{13}b_{32}b_{21}w'_l \end{pmatrix}.$$

Take any $1 \leq i < j < k \leq m$, and fix it. For any $C = (c_{ij}) \in \text{SL}(3, \mathbf{C})$, set

$$\widehat{C} := \begin{matrix} & \underline{i} & \underline{j} & \underline{k} \\ \underline{i} & \begin{pmatrix} E & O & \cdots & \cdots & O \\ O & c_{11} & c_{12} & c_{13} & \vdots \\ \vdots & c_{21} & c_{22} & c_{23} & \vdots \\ \vdots & c_{31} & c_{32} & c_{33} & O \\ O & \cdots & \cdots & O & E \end{pmatrix} \\ \underline{j} & \\ \underline{k} & \end{matrix} \in \text{SL}(m, \mathbf{C})$$

where E denotes the identity matrix. Namely, \widehat{C} is obtained from the identity matrix with (i, i) , (i, j) , $(i, k), \dots$ entries replaced with c_{11} , c_{12} , c_{13}, \dots respectively. Define the representation $\rho_7 : H \rightarrow \text{SL}(m, \mathbf{C})$ by $x_l \mapsto \widehat{C}_l$ for any $1 \leq l \leq n$. Then by considering the power series expansion with respect to z'_l and w'_l , we see that $f(\rho_7)$ can be written as a power series of z'_l and w'_l such that the minimal degree of the monomials of $f(\rho_7)$ is greater than or equal to 2. From the assumption $f \equiv 0 \pmod{(\text{mod } \mathfrak{J}_H^3)}$, we can see that each of coefficients of the monomials of degree 2 is equal to zero.

(xi). For any $p \leq q$, by observing the coefficient of $b_{12}^2 b_{23}^2 b_{31}^2 z'_p w'_q$, we see

$$\alpha_{ii,jj}(p, q) - \alpha_{ii,kk}(p, q) + \alpha_{jj,kk}(p, q) = 0.$$

In particular, by considering the case where $k = m$, we obtain $\alpha_{ii,jj}(p, q) = 0$.

(xii). For any $p \leq q$, by observing the coefficient of $b_{12}b_{13}b_{21}b_{23}b_{31}^2 z'_p w'_q$, we see

$$-\alpha_{ij,ii}(p, q) + \alpha_{ik,ki}(p, q) - \alpha_{jk,kj}(p, q) = 0.$$

Hence, if $1 < j < k$,

$$\alpha_{jk,kj}(p, q) = -\alpha_{1j,j1}(p, q) + \alpha_{1k,k1}(p, q) = 0$$

since elements type of $\bar{t}_{1i,i1}(p, q)$ does not belong to Y .

(vi) and (i). For any $p \leq q$, from the coefficients of $b_{12}^2 b_{13} b_{23} b_{31}^2 z'_p w'_q$, $b_{12} b_{13}^2 b_{21}^2 b_{32} z'_p w'_q$ and $b_{13} b_{21}^2 b_{23} b_{32}^2 z'_p w'_q$, we obtain

$$(11) \quad \alpha_{ii,ij}(p, q) - \alpha_{ij,jj}(p, q) + 2\alpha_{ij,kk}(p, q) = 0$$

$$(12) \quad -\alpha_{ii,ik}(p, q) + \alpha_{ik,kk}(p, q) + 2\alpha_{ik,jj}(p, q) = 0$$

$$(13) \quad -\alpha_{ii,ji}(p, q) + \alpha_{ji,jj}(p, q) - 2\alpha_{ji,kk}(p, q) = 0$$

respectively. By considering the case where $k = m$ in (11), we have

$$(14) \quad \alpha_{ii,ij}(p, q) = \alpha_{ij,jj}(p, q),$$

and from (11) again $\alpha_{ij,kk}(p, q) = 0$. By the same argument as above, from (13) we obtain

$$(15) \quad \alpha_{ji,kk}(p, q) = 0, \quad \alpha_{ii,ji}(p, q) = \alpha_{ji,jj}(p, q),$$

On the other hand, from (14) and (12), we have $\alpha_{ik,jj}(p, q) = 0$. Similarly, we can show

$$\alpha_{ii,jk}(p, q) = \alpha_{ii,kj}(p, q) = \alpha_{jj,ki}(p, q) = 0.$$

(x) and (vii). For any $p \leq q$, from the coefficient of $b_{12}b_{13}b_{21}^2b_{23}b_{32}z'_pw'_q$,

$$(16) \quad -3\alpha_{ik,ji}(p, q) + \alpha_{jj,jk}(p, q) + 2\alpha_{jk,kk}(p, q) = 0.$$

From this and (14), we have

$$(17) \quad \alpha_{ik,ji}(p, q) = \alpha_{jj,jk}(p, q) = \alpha_{jk,kk}(p, q).$$

Hence if $1 < j < k$,

$$\alpha_{jk,kk}(p, q) = \alpha_{1k,j1}(p, q) = 0,$$

and if $1 < k$,

$$\alpha_{1k,kk}(p, q) = \alpha_{11,1k}(p, q) = 0.$$

On the other hand, if $1 < j < k$,

$$\alpha_{jj,jk}(p, q) = \alpha_{1k,j1}(p, q) = 0.$$

(viii) and (ix). For any $p \leq q$, from the coefficient of $b_{12}b_{13}b_{23}b_{31}^2b_{32}z'_pw'_q$,

$$(18) \quad -3\alpha_{ij,ki}(p, q) + 2\alpha_{jj,kj}(p, q) + \alpha_{kj,kk}(p, q) = 0.$$

By (15), we see

$$(19) \quad \alpha_{ij,ki}(p, q) = \alpha_{jj,kj}(p, q) = \alpha_{kj,kk}(p, q).$$

Hence if $1 < j < k$,

$$\alpha_{jj,kj}(p, q) = \alpha_{kj,kk}(p, q) = \alpha_{1j,k1}(p, q) = 0,$$

and if $1 < k$,

$$\alpha_{k1,kk}(p, q) = \alpha_{11,k1}(p, q) = 0.$$

(iii). For any $p \leq q$, from (17) and (19), we have $\alpha_{ij,ki}(p, q) = \alpha_{ik,ji}(p, q) = 0$. Furthermore, by observing the coefficient of $b_{13}b_{21}b_{23}b_{31}b_{32}^2z'_pw'_q$, we obtain $\alpha_{ji,kj}(p, q) = 0$.

(iv). For any $p \leq q$, by a similar argument as above, from the coefficients of $b_{12}b_{13}b_{21}b_{23}b_{31}z'_pw'_q$, $b_{12}b_{13}^2b_{21}b_{31}b_{32}z'_pw'_q$ and $b_{12}b_{21}b_{23}b_{31}b_{32}z'_pw'_q$, we obtain

$$\alpha_{ij,jk}(p, q) = \alpha_{ik,kj}(p, q) = \alpha_{jk,ki}(p, q) = 0$$

respectively.

This completes the proof of Proposition 7.2. \square

Since $|I \setminus J| = m^2 - 1$, $t_{ij,hk}(p, q)$ generate $\Lambda^2 H_{\mathbf{Q}S}$, and $u_{ij,hk}(p, q)$ and $v_{ij}(p, q)$ generate $S^2 H_{\mathbf{Q}S}$, as a corollary to Proposition 7.2, we obtain the following.

Proposition 7.3. *For any $n \geq 2$,*

$$\mathrm{gr}^2(\mathfrak{J}_H) \cong (S^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}m^2(m^2-1)} \oplus (\Lambda^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}(m^2-1)^2(m^2-4)}.$$

7.3. The crossed homomorphism θ_H .

From the argument in Subsection 5.2, we have the crossed homomorphism

$$\tilde{\theta}_{\mathcal{D}} : \overline{\text{Aut}}(\mathfrak{J}_H/\mathfrak{J}_H^3) \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \text{gr}^2(\mathfrak{J}_H)).$$

On the other hand, $\text{Aut } F_n$ naturally acts on $\mathfrak{J}_H/\mathfrak{J}_H^3$ through the homomorphism $\text{Aut } F_n \rightarrow \text{Aut } H$. This action induces the homomorphism $\text{Aut } F_n \rightarrow \overline{\text{Aut}}(\mathfrak{J}_H/\mathfrak{J}_H^3)$. By composing these maps, we obtain the crossed homomorphism

$$\theta_H : \text{Aut } F_n \rightarrow \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \text{gr}^2(\mathfrak{J}_H)).$$

In this subsection, we study a relation between θ_H and f_M . Recall that

$$\text{gr}^1(\mathfrak{J}_H) \cong H_{\mathbf{Q}}^{\oplus m^2-1}, \quad \text{gr}^2(\mathfrak{J}_H) \cong (S^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}m^2(m^2-1)} \oplus (\Lambda^2 H_{\mathbf{Q}})^{\frac{1}{2}(m^2-1)^2(m^2-4)},$$

and that $\{s_{ij}(\overline{x}_l) \mid (i, j) \neq (m, m), 1 \leq l \leq n\}$ and Y are basis of them respectively.

By the same argument as θ_{F_n} , we can calculate the images of Nielsen's generators by θ_H . In particular, we have

$$\begin{aligned} \theta_H(S)(s_{11}(\overline{x}_1)) &= - \sum_{k=1}^m [s_{1k}(\overline{x}_1)s_{k1}(\overline{x}_1)]_3 \\ &= -\overline{v}_{11}(1, 1) - \sum_{k=2}^m \overline{u}_{1k,k1}(1, 1) \\ \theta_H(U)(s_{11}(\overline{x}_1)) &= - \sum_{k=1}^m [s_{1k}(\overline{x}_2)s_{k1}(\overline{x}_2)]_3 - \sum_{k=1}^m [s_{1k}(\overline{x}_1)s_{k1}(\overline{x}_2)]_3 \\ &= -\overline{v}_{11}(2, 2) - \sum_{k=2}^m \overline{u}_{1k,k1}(2, 2) \\ &\quad - \overline{v}_{11}(1, 2) - \frac{1}{2} \sum_{k=2}^m (\overline{u}_{1k,k1}(1, 2) + \overline{t}_{1k,k1}(1, 2)) \\ &= -(\overline{v}_{11}(2, 2) + \overline{v}_{11}(1, 2)) - \sum_{k=2}^m \left(\overline{u}_{1k,k1}(2, 2) + \frac{1}{2} \overline{u}_{1k,k1}(1, 2) \right). \end{aligned}$$

Denote $s_{ij}(\overline{x}_l)^*$ by the dual basis in $\text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \mathbf{Q})$ of $s_{ij}(\overline{x}_l)$. Let $p'_1 : \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \text{gr}^2(\mathfrak{J}_H)) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \text{gr}^2(\mathfrak{J}_H))$ be the homomorphism induced from the inclusion map $H_{\mathbf{Q}} \rightarrow \text{gr}^1(\mathfrak{J}_H)$ defined by

$$\sum_{l=1}^n c_l x_l \mapsto \sum_{l=1}^n c_l s_{11}(\overline{x}_l).$$

Let $p'_2 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \text{gr}^2(\mathfrak{J}_H)) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, S^2 H_{\mathbf{Q}})$ be the homomorphism induced from the projection $\text{gr}^2(\mathfrak{J}_H) \rightarrow S^2 H_{\mathbf{Q}}$ defined by

$$\sum_{t \in Y} c_t t \mapsto \sum_{1 \leq i \leq j \leq n} c_{\overline{v}_{11}(i,j)} x_i x_j.$$

Let $p'_3 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, S^2 H_{\mathbf{Q}}) \rightarrow \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2})$ be the homomorphism induced from the homomorphism $S^2 H_{\mathbf{Q}} \rightarrow H_{\mathbf{Q}}^{\otimes 2}$ defined by

$$x_i x_j \mapsto x_i \otimes x_j + x_j \otimes x_i$$

for any $1 \leq i \leq j \leq n$. Let $p'_4 : \text{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, H_{\mathbf{Q}}^{\otimes 2}) \cong H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} H_{\mathbf{Q}}^{\otimes 2} \rightarrow H_{\mathbf{Q}}$ be the contraction map with respect to the first and the second component. Then the composition map $p'_4 \circ p'_3 \circ p'_2 \circ p'_1$, denoted by p' , is an $\text{Aut } F_n$ -equivariant homomorphism. Set

$$f_H := p' \circ \theta_H : \text{Aut } F_n \rightarrow H_{\mathbf{Q}}.$$

Then we can see that

$$f_H(\sigma) := \begin{cases} 0, & \sigma = P, Q, \\ -x_1, & \sigma = S, \\ -x_2, & \sigma = U. \end{cases}$$

Hence, by the same argument as Theorem 6.6, we obtain the following.

Theorem 7.4. *For any $n \geq 2$,*

$$f_M = -f_H + \delta_x$$

as a crossed homomorphism. Here $x := x_1 + x_2 + \cdots + x_n \in H_{\mathbf{Q}}$.

We remark that we have

$$\begin{aligned} & H^1(\text{Aut } F_n, \text{Hom}_{\mathbf{Q}}(\text{gr}^1(\mathfrak{J}_H), \text{gr}^2(\mathfrak{J}_H))) \\ & \cong H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} S^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}m^2(m^2-1)^2} \\ & \quad \bigoplus H^1(\text{Aut } F_n, H_{\mathbf{Q}}^* \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}})^{\oplus \frac{1}{2}(m^2-1)^2(m^2-4)} \\ & \cong \mathbf{Q}^{\oplus \frac{1}{2}(m^2-1)^2(3m^2-8)}. \end{aligned}$$

8. ACKNOWLEDGMENTS

This work is supported by JSPS KAKENHI Grant Number 24740051.

REFERENCES

- [1] F. G.-Acuna; J. Maria and Montesinos-Amilibia; On the character variety of group representations in $\text{SL}(2, \mathbf{C})$ and $\text{PSL}(2, \mathbf{C})$, Math. Z. 214 (1993), 627-652.
- [2] S. Andreadakis; On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc. (3) 15 (1965), 239-268.
- [3] H. Cartan and S. Eilenberg; Homological Algebra, Princeton University Press.
- [4] M. Day; Extensions of Johnson's and Morita's homomorphisms that map to finitely generated abelian groups, Journal of Topology and Analysis, Vol. 5, No. 1 (2013), 57-85.
- [5] N. Enomoto and T. Satoh, On the derivation algebra of the free Lie algebra and trace maps. Alg. and Geom. Top., 11 (2011) 2861-2901.
- [6] R. Fricke and F. Klein; Vorlesungen über die Theorie der automorphen Functionen. Vol. 1, 365-370. Leipzig: B.G. Teuber 1897.
- [7] M. Hall; The theory of groups, second edition, AMS Chelsea Publishing 1999.
- [8] R. Hain; Infinitesimal presentations of the Torelli group, Journal of the American Mathematical Society 10 (1997), 597-651.
- [9] E. Hatakenaka and T. Satoh; On the graded quotients of the ring of Fricke characters of a free group, Journal of Algebra, 430 (2015), 94-118.
- [10] E. Hatakenaka and T. Satoh; On the rings of Fricke characters of free abelian groups, Journal of Commutative Algebra, to appear.
- [11] R. Horowitz; Characters of free groups represented in the two-dimensional special linear group, Comm. on Pure and Applied Math., Vol. XXV (1972), 635-649.
- [12] R. Horowitz; Induced automorphisms on Fricke characters of free groups, Trans. of the Amer. Math. Soc. 208 (1975), 41-50.

- [13] D. Johnson; An abelian quotient of the mapping class group, *Mathematische Annalen* 249 (1980), 225-242.
- [14] D. Johnson; The structure of the Torelli group I: A Finite Set of Generators for \mathcal{I} , *Annals of Mathematics*, 2nd Ser. 118, No. 3 (1983), 423-442.
- [15] D. Johnson; The structure of the Torelli group II: A characterization of the group generated by twists on bounding curves, *Topology*, 24, No. 2 (1985), 113-126.
- [16] D. Johnson; The structure of the Torelli group III: The abelianization of \mathcal{I} , *Topology* 24 (1985), 127-144.
- [17] N. Kawazumi; Cohomological aspects of Magnus expansions, preprint, [arXiv:math.GT/0505497](https://arxiv.org/abs/math/0505497).
- [18] W. Magnus; Über n -dimensionale Gittertransformationen, *Acta Math.* 64 (1935), 353-367.
- [19] W. Magnus; Rings of Fricke characters and automorphism groups of free groups, *Math. Z.* 170 (1980), 91-103.
- [20] W. Magnus, A. Karrass and D. Solitar; *Combinatorial group theory*, Interscience Publ., New York (1966).
- [21] S. Morita; Abelian quotients of subgroups of the mapping class group of surfaces, *Duke Mathematical Journal* 70 (1993), 699-726.
- [22] S. Morita; The extension of Johnson's homomorphism from the Torelli group to the mapping class group, *Invent. math.* 111 (1993), 197-224.
- [23] J. Nielsen; Die Isomorphismengruppe der freien Gruppen, *Math. Ann.* 91 (1924), 169-209.
- [24] T. Satoh; Twisted first homology group of the automorphism group of a free group, *J. of Pure and Appl. Alg.*, 204 (2006), 334-348.
- [25] T. Satoh; New obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group, *J. of the London Math. Soc.*, (2) 74 (2006), 341-360.
- [26] T. Satoh; The cokernel of the Johnson homomorphisms of the automorphism group of a free metabelian group, *Transactions of American Mathematical Society*, 361 (2009), 2085-2107.
- [27] T. Satoh; First cohomologies and the Johnson homomorphisms of the automorphism group of a free group. *Journal of Pure and Applied Algebra* 217 (2013), 137-152.
- [28] T. Satoh, On the lower central series of the IA-automorphism group of a free group. *J. of Pure and Appl. Alg.* 216 (2012), 709-717.
- [29] T. Satoh; A survey of the Johnson homomorphisms of the automorphism groups of free groups and related topics, to appear in *Handbook of Teichmüller theory* (A. Papadopoulos, ed.) Volume V, 167-209.
- [30] T. Satoh; The Johnson-Morita theory for the ring of Fricke characters of free groups, *Pacific Journal of Math.*, 275 (2015), 443-461.
- [31] T. Satoh; On the universal SL_2 -representation rings of free groups, *Proc. Edinb. Math. Soc.*, to appear.
- [32] T. Satoh; On the $SL(2, \mathbb{C})$ -representation rings of free abelian groups, preprint.
- [33] T. Satoh; First cohomologies and the Johnson homomorphisms of the automorphism groups of free groups II, preprint.
- [34] H. Vogt; Sur les invariants fondamentaux des équations différentielles linéaires du second ordre, *Ann. Sci. Écol. Norm. Supér.*, III. Sér. 6 (1889), 3-72.

(Takao Satoh) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE DIVISION II, TOKYO UNIVERSITY OF SCIENCE, 1-3 KAGURAZAKA, SHINJUKU, TOKYO, 162-8601, JAPAN
E-mail address: takao@rs.tus.ac.jp